Integral Methods for Transpired Boundary Layer Flow

by

Syed Fazil

A Thesis Presented to the

FACULTY OF THE COLLEGE OF GRADUATE STUDIES

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DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
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Dedicated

To

MY PARENTS

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THESIS ABSTRACT

NAME :SYED FAZIL

TITLE OF STUDY :INTEGRAL METHODS FOR TRANSPIRED BOUNDARY LAYER FLOW.

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An approximate composite one and two-parameter integral method is developed in this study for steady, two-dimensional, plane and thin axisymmetric laminar boundary layer flow with transpiration. The method operates with one-parameter for strong to moderate pressure gradient and suction, and with two-parameter for mild to adverse pressure gradient and blowing. The approach features the use of supplementary boundary layer approximations for distributions in viscous stress and velocity and the solution of the integral momentum equation and, in the two parameter mode, the integral mechanical energy equation. The method is tested for similar and nonsimilar flow with and without transpiration. The method works from separation to strong favorable pressure gradient and strong suction. The accuracy of the approach is generally within 2%. The method provides a practical, reliable and efficient approach to solving transpired laminar boundary layer flow.

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ملخص اطروحة

الاسم : سيد فاضل

عنــوان الدراسـة : طرق تكاملية لانسياب الطبقة الحدودية ذات الرشح

تاريخ نيل الدرجة : ينـاير ١٩٨٨م

تم تطوير طريقة تكاملية مركبة ذات متغير او متغيرين في هذه الدراسة بغية دراسة الاسطح ذات البعدين وانسياب الطبقة الحدودية المنتظمة مع الترشيح . تعمل هذه الطريقه بمتغير واحد عندما يكون الضغط قـويا او متوسطا وبمتغيرين عندما يكون الضغط ضعيفا او منخفضا . ويقتضي هذا الاسلوب استخدام القيم التقريبية للطبقة الحدودية المكملة للتوزيعات في حالات الضغط اللزج والسرعة والحلول التكاملية لمعادلة عزم القوى ، وبالمتغيرين في حالة معادلة الطاقة الميكانيكية التكاملية . هذا وقد تم تجريب هذه الطريقة على الانسياب المشابه او غير المشابه مع او بدون وجود الترشيح . واثبتت الطريقة جدواها في مدى يتراوح من الفصل الى الضغـط القوي والامتصاص . وتتراوح دقة هـذا الاسلـوب عمومـا في حـدود ٢ ٪ . وتقدم هذه الطريقة اسلوبا عمليا وفعـالا يمكن الاعتماد عليه لحل مشكلة الانسياب المنتظم ذو الرشح .

1. INTRODUCTION

Integral methods provide an approximate solution for boundary layer flows. These methods have long been used in tha analysis of laminar and turbulent flows. These approaches are generally relatively simple and efficient and serve as a useful complement to numerical methods in engineering analysis and design. All integral methods involve the solution of the integral momentum equation, with the more comprehensive methods also involving the solution of additional higher order integral equations.

Integral methods of basically two kinds appear in the literature. Integral methods of the first kind involve the use of boundary layer approximations for velocity or viscous stress in terms of the boundary layer thickness δ. The analysis by Pohlhausen [1] is the best known integral method of this kind. Integral methods of the second kind are based on the use of boundary layer approximations for viscous stress in terms of velocity of the type proposed by Dorodnitsyn [2].

In the integral methods of the first kind that have been developed for nontranspired flow, the velocity distribution is approximated by a polynomial of the form

$$U = \sum_{n=0}^{N} C_n \xi^n \qquad (1.1)$$

where $\xi = y/\delta$ and $U = u/U_\infty$, with U_∞ a known function of x and δ an unknown function of x. The coefficients C_n are specified in accordance with the boundary conditions and higher order equations.

A more general approximation which is applicable to transpired boundary layer flow has recently been developed for the viscous stress of the form [3]

$$\frac{\tau_{xy}}{\tau_0} = \sum_{n=0}^{N} a_n \xi^n + B_m U \tag{1.2}$$

where $B_m = \rho v_0 U_\infty/\tau_0$, v_0 is the transpiration rate and τ_0 is the wall shear stress. Here a_n is evaluated using boundary conditions and δ is considered as one of the unspecified parameter.

One-parameter integral methods of the first kind provide an efficient and simple method for analyzing laminar boundary layer flow [3]. The accuracy obtained is generally within 3 to 4% for moderate favorable pressure gradient to adverse pressure gradient. But these simple methods give considerably larger error in the vicinity of separation and breakdown for strong favorable pressure gradients. Two-parameter methods of the first kind provide quite good accuracy for adverse pressure gradient flows up to the point of separation but breakdown for strong favorable pressure gradient [4].

In the integral methods of the second kind the viscous stress has been approximated in terms of velocity by relations of the form

$$\theta = \frac{\rho U_\infty^2}{\tau_{xy}} = \frac{1}{1 - U} \sum_{j=1}^{N-1} A_j W_j(U) \qquad (1.3)$$

where A_j represents N unspecified parameters and W_j is a weighting function. Using the above approximations, many integral methods have been developed for boundary layer flow to date. But the methods have been developed in the context of rather intimidating and complicated integral equations and generally involve a fairly large number of parameters. Therefore the order and complexity of these approaches have increased over the past few years.

The primary objective of this thesis is to develop a practical, accurate and reliable integral method for transpired laminar boundary layer flow that is applicable in the important range between strong favorable pressure gradient and separation. The integral method developed will be tested for both similar and nonsimilar boundary layer flows. In addition, the present status of the integral methods for turbulent boundary layer flow will be studied and recommendations for the development of more practical approaches will be made.

2. LITERATURE SURVEY

2.1 INTRODUCTION

The aim of this chapter is to highlight the integral methods available in the literature for steady, two-dimensional, plane and thin axisymmetric laminar boundary layer flow with and without transpiration. This intuitive knowledge will provide a basis for evaluating the status of the integral methods for laminar boundary layer flow. To provide a frame of reference, both differential and integral formulations are given in this chapter.

2.2 DIFFERENTIAL FORMULATION

The differential formulation for steady, two-dimensional, plane and thin axisymmetric boundary layer flow with transpiration consists of equations for continuity and momentum in x-direction with boundary conditions which relate the dependent variables u and v. For flow in which external body forces are neglected, the differential formulation becomes

$$\rho\left[\frac{1}{r_0}\ \frac{\partial}{\partial x}(r_0 u) + \frac{\partial v}{\partial y}\right] = 0 \tag{2.1}$$

for continuity, and

$$\rho\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = \frac{\partial \tau_{xy}}{\partial y} - \frac{dP}{dx} \tag{2.2}$$

for momentum,

where $\partial\tau_{xx}/\partial x$ and dP/dy are equal to zero in accordance with the standard boundary layer approximations [5, 6].

The standard boundary conditions are of the form

$$u = 0 \quad \text{at} \quad y = 0 \tag{2.3a}$$

$$u = U_\infty \quad \text{as} \quad y \to \infty \tag{2.3b}$$

$$u = u_1 \quad \text{at} \quad x = x_1 \tag{2.3c}$$

and

$$v = v_0 \quad \text{at} \quad y = 0 \tag{2.3d}$$

where v_0 is the transpiration velocity; $v_0 > 0$ for blowing, $v_0 < 0$ for suction and $v_0 = 0$ for nontranspired flow. The viscous stress τ_{xy} is approximated by the Newton law of viscous stress which reduces to

$$\tau_{xy} = \mu \frac{\partial u}{\partial y} \tag{2.4}$$

for thin shear flows.

2.3 INTEGRAL FORMULATION

The differential formulation provides the basis for establishing useful integral equations for boundary layer flow. In the integral methods, the integrated forms of the continuity equation, Eq.(2.1), and momentum equation, Eq.(2.2), are used which provide a basis for developing simple and efficient approximate solutions for boundary layer flow. The integral momentum equation is obtained by integrating

Eq.(2.2) across the boundary layer with the continuity equation and boundary conditions appropriately incorporated. The integral momentum equation is given by [5]

$$\frac{\tau_0}{\rho U_\infty^2} = \frac{f_x}{2} = \frac{1}{r_0}\frac{d}{dx}(r_0\delta_2) + \frac{\delta_2}{U_\infty}\frac{dU_\infty}{dx}(2 + H) - \frac{v_0}{U_\infty} \qquad (2.5)$$

where r_0 is the radius of curvature, f_x is the Fanning friction factor, H is the shape factor ($\equiv \delta_1/\delta_2$), U is the dimensionless velocity ($\equiv u/U_\infty$), and the displacement thickness δ_1 and the momentum thickness δ_2 are defined by

$$\delta_1 = \int_0^\infty (1 - U)dy \qquad (2.6)$$

$$\delta_2 = \int_0^\infty U(1 - U)dy \qquad (2.7)$$

The integral momentum equation is also known as the Karman integral equation after T. Von Karman, who first suggested this approach in boundary layer analysis [5].

A useful alternate form of Eq.(2.5) is given by [5]

$$\frac{1}{r_0^2}\frac{U_\infty}{\nu}\frac{d}{dx}(r_0\delta_2)^2 = F_2 = 2[S - \lambda(2 + H) - \Omega_2] \qquad (2.8)$$

where $S = \delta_2\tau_0/(\mu U_\infty)$, $\lambda = \delta_2^2/\nu\ (dU_\infty/dx)$ and $\Omega_2 = -\delta_2 v_0/\nu$ and ν is the kinematic viscosity.

Higher order integral equations have been developed in the literature by the use of various weighting functions. For example, use of the Dorodnitsyn weighting function [2], $W_n(u) = U^n$, gives rise to a general integral equation of the form

$$\frac{1}{r_0^2}\frac{U_\infty}{\nu}\frac{d}{dx}(r_0\delta_{2n})^2 = F_{2n}$$

$$= 2H_{2n}[nU^{n-1}S + D_{1n} - \lambda H_{3n} - \Omega_2] \qquad (2.9)$$

where

$$H_{3n} = \delta_{3n}/\delta_2,\ D_{1n} = \mathcal{D}_{1n}\delta_2/(\mu U_\infty^2),\ H_{2n} = \delta_{2n}/\delta_2,$$

$$\delta_{2n} = \int_0^\infty U(1 - U^n)dy \qquad (2.10)$$

$$\delta_{3n} = \int_0^\infty [nU^{n-1}(1 - U^2) + U(1 - U^n)]dy \qquad (2.11)$$

and

$$\mathcal{D}_{1n} = \int_0^\infty n(n-1)U^{n-2}\tau_{xy}\frac{\partial u}{\partial y}dy \qquad (2.12)$$

Equation (2.9) reduces to the integral momentum equation, Eq.(2.8) for n = 1, and to the integral mechanical energy equation,

$$\frac{1}{r_0^2}\frac{U_\infty}{\nu}\frac{d}{dx}(r_0\delta_3)^2 = F_3 = 2H_{32}[D_1 - 3\lambda H_{32} - \Omega_2] \qquad (2.13)$$

for n = 2, where $F_3 = F_{22}$, $\delta_3 = \delta_{22}$, $H_{32} = \delta_3/\delta_2$, $D_1 = \mathcal{D}_1\delta_2/(\mu U_\infty^2)$

$$\delta_3 = \int_0^\infty U(1 - U^2)dy \qquad (2.14)$$

$$\mathcal{D}_1 = \int_0^\infty \tau_{xy} \frac{\partial u}{\partial y} dy \qquad (2.15)$$

and to

$$\frac{1}{r_0^2} \frac{U_\infty}{\nu} \frac{d}{dx}(r_0 \delta_4)^2 = F_4 = 2H_{42}[D_2 - \lambda(4H_{42} - 3) - \Omega_2] \qquad (2.16)$$

for n = 3, where $F_4 = F_{23}$, $\delta_4 = \delta_{23}$, $H_{42} = \delta_4/\delta_2$, $D_2 = \mathcal{D}_2\delta_2/(\mu U_\infty^2)$

$$\delta_4 = \int_0^\infty U(1 - U^3)dy \qquad (2.17)$$

$$\mathcal{D}_2 = \int_0^\infty \tau_{xy} U \frac{\partial u}{\partial y} dy \qquad (2.18)$$

The integral mechanical energy equation, Eq.(2.13) can be obtained by multiplying the momentum equation by the velocity u, which converts forces into the rate of work done by those forces, hence the name mechanical energy. It represents the balance of mechanical energy within a small section of the boundary layer. This equation was first derived by Leibenson [31] and later independently by Wieghardt [32].

2.3.1 Similar Boundary Layer Flows

Similar transpired boundary layer flows are characterized by distributions in the transpiration rate v_0 and the free stream velocity U_∞ of the forms [6]

$$v_0 \propto x^{(m-1)/2} \qquad (2.19)$$

$$U_\infty = C\ x^m \tag{2.20}$$

where C and m are both constants. The constant m is known as the pressure gradient parameter and is defined by

$$m = \frac{x}{U_\infty}\frac{dU_\infty}{dx} \tag{2.21}$$

Equation (2.19) can also be written as

$$\frac{v_0}{U_\infty} \propto \frac{1}{\sqrt{Re_x}} \tag{2.22}$$

The blowing parameter BP,

$$BP = \frac{v_0}{U_\infty}\sqrt{Re_x} \tag{2.23}$$

is also constant for similar flows. In addition, similar boundary layer flows are characterized by constant values of the integral parameters λ, S, H, Ω, Ω_2, F_2, F_3, Λ and associated integral parameters.

2.3.2 Nonsimilar Boundary Layer Flows

Nonsimilar boundary layer flows are characterized by distributions in v_0 and U_∞ that do not satisfy Eqs.(2.19) and (2.20) and by non-uniform distributions in the integral parameters S, H, H_{32}, F_2, F_3, λ, Ω, and Ω_2. Because of the variation of F_2 and F_3 with x, the integral momentum equation and the integral mechanical energy equation cannot be integrated to obtain an analytical solution. However, approximate integral solutions for nonsimilar boundary layer flows can be obtained by use of simple numerical finite difference methods.

2.4 INTEGRAL METHODS

Integral methods appearing in the literature can be classified according to the type and number of parameters that are employed in approximating U or τ_{xy}. The most well known type of integral methods (first kind) involves the use of approximations for U (or τ_{xy}) in terms of y, with the boundary layer thickness δ generally used as a primary parameter. Another important type of integral method (second kind) involves the use of approximations for viscous stress τ_{xy} in terms of velocity U. The evaluation of the unspecified parameter is accomplished in integral methods of both kinds by solving one or more integral equations. One-parameter methods involve the solution of one integral equation (usually the integral momentum equation). Two-parameter methods require the solution of two integral equations, and so forth. Background on integral methods of both the first and second kinds is presented in this section.

2.4.1 Integral Methods of the First Kind

Boundary layer approximations of the form

$$U = \sum_{n=1}^{N+1} C_n \left(\frac{y}{\delta}\right)^n = \sum_{n=1}^{N+1} C_n \xi^n \tag{2.24}$$

Where $\xi = y/\delta$ have been traditionally used in the development of integral solutions for many years. This type of approximation was featured in the early one-parameter integral methods by Pohlhausen

[1], Timman [8], Mangler [9], and others. The coefficients C_n for one-parameter methods of this type are simply established on the basis of boundary conditions. For example, with the conditions $u|_{y=0} = 0$, $u|_{y=\delta} = U_\infty$, $\frac{\partial u}{\partial y}|_{y=\delta} = 0$, $\frac{\partial^2 u}{\partial y^2}|_{y=\delta} = 0$ and $\mu\frac{\partial^2 u}{\partial y^2}|_{y=0} = \frac{dP}{dx}$ satisfied, Eq.(2.24) reduces to the famous Pohlhausen polynomial approximation,

$$\frac{u}{U_\infty} = 2\xi - 2\xi^3 + \xi^4 + \frac{\Lambda}{6}\,\xi(1 - \xi)^3 \qquad (2.25)$$

Where $\Lambda = \delta^2/\nu\ (dU_\infty/dx)$. Multiple parameter integral methods of this kind can be developed by leaving one or more of the coefficients unspecified.

Boundary layer approximations of the form of Eq.(2.24) suffer from the disadvantage that it cannot be effectively used for transpired laminar boundary layer flow and turbulent boundary layer flow. In order to overcome this problem, a more general and widely applicable approximation for incompressible transpired boundary layer flow has recently been developed by the use of supplementary boundary layer approximations for viscous stress [3].

Other integral methods of the first kind have been proposed which make use of relations for the integral parameters which are based on the Falkner-Skan family of similar flows. Early formulations of this type were developed by Waltz [10] and Truckenbrodt [11].

More recent developments along this line have been reported by [12]. Integral methods of this type do not break down for strong favorable pressure gradients. However, they do not provide a basis for generalization.

2.4.1.1 Concept of Supplementary Boundary Layer Approximations

In addition to standard boundary layer approximations, supplementary boundary layer approximations can be used in the analysis of boundary layer flows. The approximation for the distribution in dimensionless shear stress τ_{xy}/τ_0 constitutes a supplementary boundary layer approximation. The best known boundary layer approximation of this kind is the Couette law [5, 7],

$$\frac{\tau_{xy}}{\tau_0} = 1 + \beta_\delta \xi + B_m U \tag{2.26}$$

where $\beta_\delta = \delta/\tau_0(dP/dx)$ and $B_m = \rho v_0 U_\infty/\tau_0$. Using supplementary boundary layer approximations for τ_{xy}/τ_0, useful approximations for the distribution in the dimensionless velocity U can be obtained which can be employed in the development of integral solution methods.

Following the approach presented in [3], to formulate a two-parameter integral method for transpired laminar boundary layer flow, approximations are developed for the distribution in viscous stress and velocity. In this approach, the distribution in viscous stress τ_{xy} should satisfy the Couette law, Eq.(2.26), within the close vicinity of the wall, the physical constraints

$$\frac{\partial \tau_{xy}}{\partial y} = 0 \quad \text{and/or} \quad \tau_{xy} = 0 \quad \text{at} \quad y = \delta \tag{2.27a}$$

$$\frac{\partial U}{\partial y} = 0 \quad \text{and} \quad U = 1 \qquad \text{at } y = \delta \tag{2.27b}$$

at the outer edge of the boundary layer, and higher order conditions. An Nth order polynomial type approximation that satisfies these requirements is given by

$$\frac{\tau_{xy}}{\tau_0} = \sum_{n=0}^{N} a_n \xi^n \tag{2.28a}$$

for nontranspired flow, and

$$\frac{\tau_{xy}}{\tau_0} = \sum_{n=0}^{N} a_n \xi^n + B_m U \tag{2.28b}$$

for transpired flow.

To illustrate, a 4th order approximation for τ_{xy}/τ_0 is obtained by setting $N = 4$; that is,

$$\frac{\tau_{xy}}{\tau_0} = \sum_{n=0}^{4} a_n \xi^n + B_m U \tag{2.29}$$

The Couette law is satisfied by requiring $a_0 = 1$ and $a_1 = \beta_\delta$. Using the physical constraints given by Eq.(2.27), a_2 and a_3 are evaluated in terms of a_4.

$$\frac{\tau_{xy}}{\tau_0} = 1 + \beta_\delta \xi + B_m U - (3 + 2\beta_\delta + 3B_m)\xi^2 + (2 + \beta_\delta + 2B_m)\xi^3$$

$$+ a_4(\xi^2 - 2\xi^3 + \xi^4) \tag{2.30}$$

Here δ and a_4 are the two unspecified parameters.

Equation (2.28) is multiplied by Mo_δ to put it into the convenient dimensionless form

$$\frac{\tau_{xy}\delta}{\mu U_\infty} = Mo_\delta \sum_{n=0}^{N} a_n \xi^n - \Omega U \tag{2.31}$$

This equation can also be written as

$$\frac{\tau_{xy}\delta}{\mu U_\infty} = Mo_\delta \sum_{n=0}^{N} \alpha_n \xi^n + \Lambda \sum_{n=0}^{N} \beta_n \xi^n + \Omega \sum_{n=0}^{N} \gamma_n \xi^n - \Omega U \tag{2.32}$$

Where

$$a_n = \alpha_n - \beta_n \beta_\delta - \gamma_n B_m; \quad Mo_\delta = \tau_0 \delta/(\mu U_\infty),$$

$$\Lambda = \delta^2/\nu\ (dP/dx) = -\ Mo_\delta \beta_\delta, \quad \Omega = -v_0 \delta/\nu = -\ Mo_\delta B_m$$

The coefficients α_n, β_n, γ_n for two-parameter method are listed in Table. 2-1 for N = 4.

The Newton law of viscous stress, Eq.(2.4), is used to develop a relationship for the velocity distribution; that is,

$$\tau_{xy} = \mu \frac{\partial u}{\partial y}$$

or

$$\frac{\tau_{xy}\delta}{\mu U_\infty} = \frac{dU}{d\xi} \tag{2.33}$$

Combining Eq.(2.31) and Eq.(2.33) gives

Table 2-1 Coefficients α_n, β_n and γ_n for N = 4 (Two-parameter method).

α_0	α_1	α_2	α_3	α_4	β_0	β_1	β_2	β_3	β_4	γ_0	γ_1	γ_2	γ_3	γ_4
1	0	$-3+\alpha_4$	$2-2\alpha_4$	α_4	0	-1	$2+\beta_4$	$-1-2\beta_4$	β_4	0	0	$3+\gamma_4$	$-2-2\gamma_4$	γ_4

$$\frac{dU}{d\xi} + \Omega U = Mo_\delta \sum_{n=0}^{N} a_n \xi^n \tag{2.34}$$

This differential equation has been solved by using the integration factor $e^{\Omega\xi}$ and the boundary condition U = 0 at ξ = 0. The solution is given by [3]

$$U = \sum_{n=0}^{N} C_n \xi^n - C_0 e^{-\Omega\xi} \tag{2.35}$$

where

$$C_n = Mo_\delta \sum_{m=n}^{N} a_m J_{nm}(\Omega) \text{ and } J_{nm}(\Omega) = \frac{(-1)^{m-n} m!}{\Omega^{m-n+1} n!}$$

A relationship is developed between Mo_δ and the parameters Λ and Ω by using Eq.(2.35) and the boundary condition, U = 1 at ξ = 1. The resulting relation for Mo_δ is given by [3]

$$Mo_\delta = \frac{M_0 + M_1 e^{-\Omega}}{M_2 + M_3 e^{-\Omega}} \tag{2.36}$$

where

$$M_0 = 1 - \sum_{n=0}^{N} \sum_{m=n}^{N} (\beta_m \Lambda + \gamma_m \Omega) J_{nm}(\Omega)$$

$$M_1 = - \sum_{n=0}^{N} (\beta_n \Lambda + \gamma_n \Omega) J_n(\Omega)$$

$$M_2 = \sum_{n=0}^{N} \sum_{m=n}^{N} \alpha_m J_{nm}(\Omega)$$

$$M_3 = \sum_{n=0}^{N} \alpha_n J_n(\Omega)$$

For nontranspired boundary layer flow ($\Omega = 0$), the solution to Eq.(2.34) can be represented by Eq.(2.24),

$$U = \sum_{n=1}^{N+1} C_n \xi^n \tag{2.24}$$

where $C_n = b_n + \Lambda c_n$, $b_n = \dfrac{\alpha_{n-1}}{n} b_1$ and $c_n = \dfrac{\alpha_{n-1}}{n} c_1 + \dfrac{\beta_{n-1}}{n}$

The corresponding relation for Mo_δ is given by

$$Mo_\delta = b_1 + \Lambda c_1 \tag{2.37}$$

where $b_1 = \dfrac{1}{\sum_{n=0}^{N} \dfrac{\alpha_n}{n+1}}$ and $c_1 = -\dfrac{\sum_{n=0}^{N} \dfrac{\beta_n}{n+1}}{\sum_{n=0}^{N} \dfrac{\alpha_n}{n+1}}$

2.4.1.2 Integral Parameters

The pertinent integral relations associated with the use of boundary layer approximations of the first kind for transpired and nontranspired laminar boundary layer flows are given by

$$\frac{\delta_1}{\delta} = \int_0^1 (1 - U)d\xi \qquad \frac{\delta_2}{\delta} = \int_0^1 U(1 - U)d\xi \tag{2.38a,b}$$

$$\frac{\delta_3}{\delta} = \int_0^1 U(1 - U^2)d\xi \qquad \frac{\delta_4}{\delta} = \int_0^1 U(1 - U^3)d\xi \tag{2.38c,d}$$

$$D_1 = \frac{\delta_2}{\delta}\int_0^1 \left(\frac{\tau_{xy}\delta}{\mu U_\infty}\right)^2 d\xi \qquad S = \frac{\delta_2}{\delta} Mo_\delta \tag{2.39a,b}$$

$$H = \frac{\delta_1/\delta}{\delta_2/\delta} \qquad H_{32} = \frac{\delta_3/\delta}{\delta_2/\delta} \tag{2.39c,d}$$

$$\lambda = (\delta_2/\delta)^2\Lambda \qquad \Omega_2 = (\delta_2/\delta)\Omega \qquad (2.39e,f)$$

The integral relations associated with the integral method of the first kind are listed in Tables. 2-2 and 2-3 for transpired and nontranspired boundary layers respectively.

These results provide a basis for developing one and two-parameter integral solutions for similar and nonsimilar laminar boundary layer flows.

2.4.1.3 Integral Solutions for Similar Boundary Layer Flows

With the distribution in U_∞ given by Eq.(2.20) and the integral parameters λ, S, H, Ω, Ω_2, F_2, F_3, Λ held constant for similar plane boundary layer flow, the solutions to the integral momentum equation, Eq.(2.8), and the integral mechanical energy equation, Eq.(2.13), are given by [3]

$$\delta_2^2 = \frac{F_2}{1 - m}\frac{\nu x}{U_\infty} \qquad (2.40)$$

and

$$H_{32}^2 F_2 = F_3 \qquad (2.41)$$

Using Eq.(2.40), the parameters λ and Ω_2 are expressed in terms of m and v_0/U_∞ by writing

$$\lambda = \frac{\delta_2^2}{\nu}\frac{dU_\infty}{dx} = \frac{m}{1 - m}F_2 \qquad (2.42)$$

Table 2-2 Integral relations for the integral method of the first kind : Transpired boundary layer flow.

$$U = \sum_{n=0}^{N} C_n \xi^n - C_0 e^{-\Omega\xi}$$

$$\frac{dU}{d\xi} = \frac{\tau_{xy}\delta}{\mu U_\infty} = \sum_{n=1}^{N} n C_n \xi^{n-1} + \frac{C_0 e^{-\Omega\xi}}{\Omega}$$

$$\frac{\delta_1}{\delta} = 1 - \sum_{n=0}^{N} \frac{C_n}{n+1} - \frac{C_0}{\Omega}\left(e^{-\Omega} - 1\right)$$

$$\frac{\delta_2}{\delta} = \sum_{n=0}^{N} \frac{C_n}{n+1} + \frac{C_0}{\Omega}\left(e^{-\Omega} - 1\right) + \frac{C_0^2}{2\Omega}\left(e^{-2\Omega} - 1\right)$$

$$+ 2C_0 \sum_{n=0}^{N} C_n J_n(-\Omega) - \sum_{n=0}^{N} \sum_{i=0}^{N} \frac{C_n C_i}{n+i+1}$$

where

$$J_0(-\Omega) = \int_0^1 e^{-\Omega\xi} d\xi = \frac{e^{-\Omega} - 1}{-\Omega}$$

$$J_n(-\Omega) = \int_0^1 \xi^n e^{-\Omega\xi} d\xi = \frac{e^{-\Omega}}{-\Omega} - \frac{n}{-\Omega} J_{n-1}(-\Omega)$$

$$\frac{\delta_3}{\delta} = \sum_{n=0}^{N} \frac{C_n}{n+1} + \frac{C_0}{\Omega}\left(e^{-\Omega} - 1\right) - \frac{C_0^3}{3\Omega}\left(e^{-3\Omega} - 1\right)$$

$$+ 3C_0 \sum_{n=0}^{N} \sum_{i=0}^{N} C_n C_i J_{ni}(-\Omega) - 3C_0^2 \sum_{n=0}^{N} C_n J_n(-2\Omega)$$

$$- \sum_{n=0}^{N} \sum_{i=0}^{N} \sum_{j=0}^{N} \frac{C_n C_i C_j}{n+i+j+1}$$

where

$$J_{ni}(-\Omega) = \int_0^1 \xi^{n+1} e^{-\Omega\xi} d\xi$$

$$= \int_0^1 [\sum_{n=0}^{N} nC_n \xi^{n-1} + \frac{C_0 e^{-\Omega\xi}}{\Omega}]^2 d\xi$$

$$D_1 = \frac{\delta_2}{\delta}[\sum_{n=1}^{N} \sum_{i=1}^{N} \frac{(nC_n)(iC_i)}{n + i - 1} + 2C_0\Omega \sum_{n=1}^{N} nC_n J_{n-1}(-\Omega)$$

$$- \frac{C_0^2\Omega^2}{2\Omega}(e^{-2\Omega} - 1)]$$

Table 2-3 Integral relations for the integral method of the first kind: Nontranspired boundary layer flow.

$$U = \sum_{n=1}^{N+1} C_n \xi^n$$

$$\frac{dU}{d\xi} = \frac{\tau_{xy}\delta}{\mu U_\infty} = \sum_{n=1}^{N+1} nC_n \xi^{n-1}$$

$$\frac{\delta_1}{\delta} = 1 - \sum_{n=1}^{N+1} \frac{C_n}{n+1}$$

$$\frac{\delta_2}{\delta} = \sum_{n=1}^{N+1} \frac{C_n}{n+1} - \sum_{n=1}^{N+1} \sum_{i=1}^{N+1} \frac{C_n C_i}{n+i+1}$$

$$\frac{\delta_3}{\delta} = \sum_{n=1}^{N+1} \frac{C_n}{n+1} - \sum_{n=1}^{N+1} \sum_{i=1}^{N+1} \sum_{j=1}^{N+1} \frac{C_n C_i C_j}{n+i+j+1}$$

$$D_1 = \frac{\delta_2}{\delta}\left[\sum_{n=1}^{N+1} \sum_{i=1}^{N+1} \frac{(nC_n)(iC_i)}{n+i-1}\right]$$

and

$$\Omega_2 = -\frac{v_0 \delta_2}{\nu} = -\frac{v_0}{U_\infty}\sqrt{Re_x}\,\sqrt{F_2/(1 - m)} \tag{2.43}$$

and the blowing parameter BP is put into the form

$$BP = \frac{v_0}{U_\infty}\sqrt{Re_x} = -\frac{\Omega_2}{\sqrt{\lambda + F_2}} \tag{2.44}$$

The pressure gradient parameter m is written as

$$m = \frac{\lambda}{\lambda + F_2} \tag{2.45}$$

The Falkner-Skan acceleration parameter β is expressed in terms of m by [5, 7]

$$\beta = \frac{2m}{m + 1} = \frac{\lambda}{2\lambda + F_2} \tag{2.46}$$

and the friction factor is given by

$$\frac{f_x}{2}\sqrt{Re_x} = \frac{S}{\sqrt{\lambda + F_2}} \tag{2.47}$$

Equation (2.41) provides a second independent relation which can be used in two-parameter approaches. This equation is represented in the residual form,

$$R = 1 - \frac{[H_{32}(S + \lambda(1 - H)) + \Omega_2(1 - H_{32})]}{2D_1} \tag{2.48}$$

where R = 0.

Similar plane stagnation flows are characterized by $F_2 = 0$ and m = 1. For similar axisymmetric stagnation flow, the radius of curvature

r_0 is equal to x and U_∞ is given by Eq.(2.20) with m = 1 and the solution to integral momentum equation, Eq.(2.8) takes the form

$$\delta_2^2 = \frac{F_2}{2}\frac{\nu x}{U_\infty} \tag{2.49}$$

The solution to the integral mechanical energy equation for this case is represented by Eq.(2.41). Equation (2.42) leads to

$$\lambda = \frac{F_2}{2}m = \frac{F_2}{2} \tag{2.50a}$$

or

$$(2\lambda - F_2) = 0 \tag{2.50b}$$

The blowing parameter

$$BP = -\sqrt{2/F_2}\ \Omega_2 \tag{2.51}$$

and friction factor

$$\frac{f_x}{2}\sqrt{Re_x} = \sqrt{2/F_2}\ S \tag{2.52}$$

The above relations can be used in one and two-parameter integral methods to develop approximate solutions for the velocity distribution and friction factor $(f_x/2)\sqrt{Re_x}$ as a function of m or β for similar boundary layer flow.

2.4.1.4 Integral Solutions for Nonsimilar Boundary Layer Flows.

In order to obtain numerical solutions for plane and thin axisymmetric nonsimilar boundary layer flows the integral momentum equation, Eq.(2.8) is written as [3].

$$\frac{U_\infty}{\nu}\frac{d\delta}{dx/L} = G_2 \qquad (2.53)$$

where

$$G_2 = (\frac{\delta}{\delta_2})^2 [\frac{F_2}{2} - \frac{\delta_2}{L}(\frac{Re_{\delta_2}}{r_0}\frac{dr_0}{dx/L} + Re_\delta \frac{d}{dx/L}(\frac{\delta_2}{\delta}))]$$

Rearranging, this equation is put into the form

$$Re_{\delta_{i+1}} = \frac{U_{\infty_{i+1}}}{U_{\infty_i}}(Re_\delta + \frac{L}{\delta}G_2\frac{\Delta x}{L})_i \qquad (2.54)$$

where i = 1,2,3........ The parameters Λ and Ω are expressed in terms of Re_δ by

$$\Lambda = \frac{Re_\delta^2\, m}{Re_x} \qquad (2.55)$$

and

$$\Omega = -\frac{v_0}{U_\infty}Re_\delta \qquad (2.56)$$

The parameters Λ and Ω can be calculated at station i+1.

The integral mechanical energy equation, Eq.(2.13), for nonsimilar boundary layer flow can be written as

$$H_{32}^2 F_2 + 2H_{32}\frac{Re_\delta^2}{(U_\infty L/\nu)}\frac{dH_{32}}{dx/L} = F_3 \qquad (2.57)$$

or alternatively in the residual form

$$R = 1 - [H_{32}(S + \lambda(1 - H)) + \frac{\lambda}{m}\frac{x}{L}\frac{dH_{32}}{dx/L}$$
$$+ \Omega_2(1 - H_{32})]/2D_1 \qquad (2.58)$$

The friction factor is expressed in terms of Re_δ by

$$\frac{f_x}{2} = \frac{S}{(\delta_2/\delta)} Re_\delta \qquad (2.59)$$

The above relations can be used in one and two-parameter integral methods of the first kind to develop simple numerical finite difference methods for analyzing nonsimilar boundary layer flows.

2.4.1.5 One-Parameter Methods

The boundary layer thickness δ is the only parameter used in standard one-parameter methods of the first kind. All the coefficients are specified in accordance with the boundary conditions. The solution for δ is generally obtained by using the integral momentum equation, Eq.(2.8). One-parameter integral methods of the first kind have been developed for nontranspired laminar boundary layer flow by Pohlhausen [1], Timman [8], Mangler [9]. All these methods fail for strong favorable pressure gradients, and give rise to large errors in the vicinity of separation. Furthermore, these methods do not provide a basis for generalization, and hence cannot be used for transpired boundary layer flows and turbulent boundary layer flows. However, a one-parameter method using second and third order approximations for viscous stress (refer to section 2.4.1.1) has been recently developed for transpired laminar boundary layer flow by Thomas and Amminger [3]. Although this method applies in the region from stagnation to separation and can be extended to turbulent flows, it breaks down for moderately strong favorable pressure gradient and suction. The accuracy of the method is generally about 3%, except in the vicinity of separation where the error can boost from 10 to 15%.

2.4.1.6 Two-Parameter Methods

The boundary layer thickness δ and an additional parameter (such as a coefficient C_{N+1} or a_N) are featured in the standard type two-parameter integral methods of the first kind, with closure generally accomplished by the solution of the integral momentum equation, Eq.(2.8), and integral mechanical energy equation, Eq.(2.13). A two-parameter integral method of the first kind has been developed for nontranspired flow by Wieghardt [13]. The famous Wieghardt integral method features the use of a 12th order approximation for U of the form

$$U = \sum_{n=0}^{12} C_n \xi^n \tag{2.60}$$

with 12 of the 13 coefficients C_0, C_1, C_2........, C_{12} evaluated on the basis of the following 12 boundary constraints.

$$u = 0 \quad \text{at} \quad y = 0 \tag{2.61a}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{dP}{dx} \text{ at } y = 0 \tag{2.61b}$$

$$\frac{\partial^3 u}{\partial y^3} = 0 \text{ at } y = 0 \tag{2.61c}$$

and

$$U=1,\ \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} = \dots\dots\dots\frac{\partial^8 u}{\partial y^8} = 0 \text{ at } y = \delta \tag{2.61d}$$

Using Eqs.(2.61a) to (2.61d) to evaluate 12 of the coefficients, Wieghardt presented a relation for U of the form

$$U = f_1(\xi) + af_2(\xi) + bf_3(\xi) \tag{2.62}$$

with

$$f_1(\xi) = 1 - (1 - \xi)^8(1 + 8\xi + 36\xi^2 + 120\xi^3)$$

$$f_2(\xi) = (1 - \xi)^8\xi(1 + 8\xi + 36\xi^2)$$

$$f_3(\xi) = -(1 - \xi)^8\xi^2(1 + 8\xi)$$

where a and b represent the two-parameters to be evaluated. In this connection, b is related to δ by

$$b = \frac{\Lambda}{2} = \frac{\delta^2}{2\nu}\frac{dU_\infty}{dx} \tag{2.63}$$

such that the two parameters could be considered to be δ and a.

To simplify matters, Tani [14] proposed a 4th order approximation for U of the form

$$U = \sum_{n=0}^{4} C_n\xi^n \tag{2.64}$$

where 4 of the 5 coefficients satisfy the 4 conditions

$$u = 0 \text{ at } y = 0 \tag{2.65a}$$

and

$$u = U_\infty, \; \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} = 0 \text{ at } y = \delta \tag{2.65b,c,d}$$

However, it should be noted that the important boundary condition represented by Eq.(2.61b) which is related to the Couette law is not satisfied in this formulation. The relation for U proposed by Tani is of the form

$$U = (6 - 8\xi + 3\xi^2)\xi^2 + a\xi(1 - \xi)^3 \tag{2.66}$$

where a and δ are the two parameters.

Using the supplementary boundary layer approximation for τ_{xy} given by Eq.(2.28), a more general type of two-parameter integral method of the first kind has been developed by Thomas and Amminger [4] for nontranspired boundary layer flow. This method ascertains good accuracy up to separation and provides a basis for generalization to transpired and turbulent flows, but fails for strong favorable pressure gradient.

In Truckenbrodt's [11] method the relations between S, H, H_{32} and D_1 are the same as those which hold for the family of similar profiles.

To summarize, all of the two-parameter integral methods of the first kind which employ polynomial approximations for U or τ_{xy} fail for strong favorable pressure gradient because of velocity overshoot. The method by Wieghardt (12th order) and the stress method [3,4]

(4th order) both satisfy the Couette law in the region near the wall. These methods prove to be quite accurate. On the other hand, the method by Tani does not require that the Couette law be satisfied. As a result, the accuracy associated with this method is significantly less. For example, the errors in solution results for separating similar flow are of the order of 10%. Methods such as the one developed by Truckenbrodt which involve the use of similarity solutions do not breakdown for strongly accelerated flow, but lack sufficient generality to be readily extended to transpiration and turbulence. Because of its relative simplicity and general nature, the stress method [3, 4] would appear to be the best candidate for further development, indicating that the breakdown associated with favorable pressure gradient can be overcome.

2.4.2 Integral Methods of the Second Kind

The most commonly used boundary layer approximations of the second kind are of the form suggested by Dorodnitsyn [2].

$$\theta = \frac{\rho U_\infty^2}{\tau_{xy}} = \frac{1}{1-U}\sum_{n=0}^{N-1} A_n U^n \tag{2.67}$$

or

$$\frac{\theta}{\theta_0} = \frac{\tau_0}{\tau_{xy}} = \frac{1 + \sum\limits_{n=0}^{N-1} \alpha_n U^n}{1-U} \tag{2.68}$$

where $\alpha_n = A_n/A_0$ and $A_0 = \theta_0 = \rho U_\infty^2/\tau_0 = \sqrt{2/f_x}$.

Equation (2.67) satisfies the primary boundary conditions $\tau_{xy} = \tau_0$ at $y = 0$ where $U = 0$, and $\tau_{xy} = 0$ as $U \to 1$ for applications in which the free stream velocity U_∞ is not zero. The coefficients $A_0, A_1, A_2, \ldots\ldots, A_{n-1}$ must be established by the use of N-1 integral equations. With the stress represented by Eq.(2.65), the velocity distribution is obtained from

$$\frac{yU_\infty}{\nu} = \int_0^U \theta dU \tag{2.69}$$

Whereas one-parameter methods of this kind which involve the coefficient A_0 are not sufficiently accurate, two-parameter and multi-parameter methods of this kind have been extensively developed for a wide span of nonseparating flows [15].

An alternate approximation of the second kind has been developed for nonseparating flows which is of the form [2]

$$\theta = \frac{\sum_{n=1}^{N-1} C_n U^n}{(1 - U)\sqrt{C_0 + U}} \tag{2.70}$$

However, this method is rather awkward and has not been widely used.

It should be emphasized that the integral methods of the second kind have been developed in the context of rather intimidating and needlessly complicated integral equations of the form given by Dorodnitsyn [2] for nontranspired flows,

$$\frac{d}{d\xi}\int_0^\infty Uf(U)d\eta = \frac{\dot{U}_\infty}{U_\infty}\int_0^\infty (1 - U^2)d\eta - f'(0)\frac{\partial U}{\partial \eta}\Big|_{\eta=0}$$

$$- \int_0^\infty \left(\frac{\partial u}{\partial \eta}\right)^2 f''(U)d\eta \tag{2.71}$$

where

$$\xi = \int_0^x U_\infty dx$$

$$\eta = \frac{1}{\sqrt{v}}\int_0^y U_\infty dy = \frac{U_\infty y}{\sqrt{v}}$$

However, the simple integral momentum, integral mechanical energy and higher order integral equations given by Eqs.(2.8), (2.13) and (2.16) are just as effective.

To incorporate boundary layer approximations of the second kind into the simple system of integral equations given by Eqs.(2.8), (2.13) and (2.16), the integral relations are put into the form

$$\frac{\delta_1 U_\infty}{v} = \int_0^1 \theta(1 - U)dU \tag{2.72}$$

$$\frac{\delta_2 U_\infty}{v} = \int_0^1 \theta U(1 - U)dU \tag{2.73}$$

$$\frac{\delta_3 U_\infty}{v} = \int_0^1 \theta U(1 - U^2)dU \tag{2.74}$$

$$\frac{\delta_4 U_\infty}{v} = \int_0^1 \theta U(1 - U^3)dU \tag{2.75}$$

$$\frac{\mathcal{D}_1}{\rho U_\infty^3} = \int_0^1 \frac{1}{\theta} dU \tag{2.76}$$

$$\frac{\mathcal{D}_2}{\rho U_\infty^3} = \int_0^1 \frac{U}{\theta} dU \tag{2.77}$$

$$D_1 = \frac{\mathcal{D}_1}{\rho U_\infty^3} \frac{\delta_2 U_\infty}{\nu} \tag{2.78}$$

$$D_2 = \frac{\mathcal{D}_2}{\rho U_\infty^3} \frac{\delta_2 U_\infty}{\nu} \tag{2.79}$$

Practical two and three-parameter forms of Eq.(2.65) are given by

$$\theta = \frac{A_0(1 + \alpha_{10} U)}{1 - U} \tag{2.80}$$

and

$$\theta = \frac{A_0(1 + \alpha_{10} U + \alpha_{20} U^2)}{1 - U} \tag{2.81}$$

The two and three-parameter integral methods of the second kind provide good accuracy for mild adverse to strong favorable pressure gradients and suction but break for mild to moderate adverse pressure gradient.

Multi-parameter integral method of the second kind can be developed by using the approximation of θ given by Eq.(2.67). The approach can be extended to near separating flows by the use of larger number of parameters. However, this proves to be rather impracticable. Because multi-parameter formulations such as these become increasingly complicated as the order of the approximation is raised, this approach has not been carried out in the literature beyond the four-parameter level for boundary layers. However, higher order formulations (with N as large as 14) have been developed using orthonormal weighting functions in developing the integral equations and approximations for the shear stress. (Fletcher and Holt).

3. INTEGRAL METHOD FOR LAMINAR TRANSPIRED BOUNDARY LAYER FLOW

3.1 INTRODUCTION

In this chapter strong emphasis is placed on the development of an integral method for laminar boundary layer flow with and without transpiration. The method should be capable of handling the flow with strong favorable pressure gradient. To achieve this, both one and two-parameter integral methods of the first kind have been considered. The results obtained by the present method for similar and nonsimilar flow with and without transpiration are also discussed.

3.2 EXAMINATION OF ONE AND TWO-PARAMETER INTEGRAL METHODS OF THE FIRST KIND FOR STRONG FAVORABLE PRESSURE GRADIENTS

The two-parameter integral methods of Wieghardt [13], Tani [14] and Thomas and Amminger [4] indicate velocity popping for strong favorable pressure gradient. Consequently, the method breaks down. For example, the 4th order two-parameter stress method [4] breaks down for similar flow with $\Lambda > 5$, as depicted in Fig. 3-1. To pursue this matter further, this general approach was considered for higher order (i.e., N = 5, 6, 7) polynomial approximations to determine whether a two-parameter method of this kind could be developed for strong favorable pressure gradient. However, like the 12th order method of Wieghardt, these higher order approximations result in

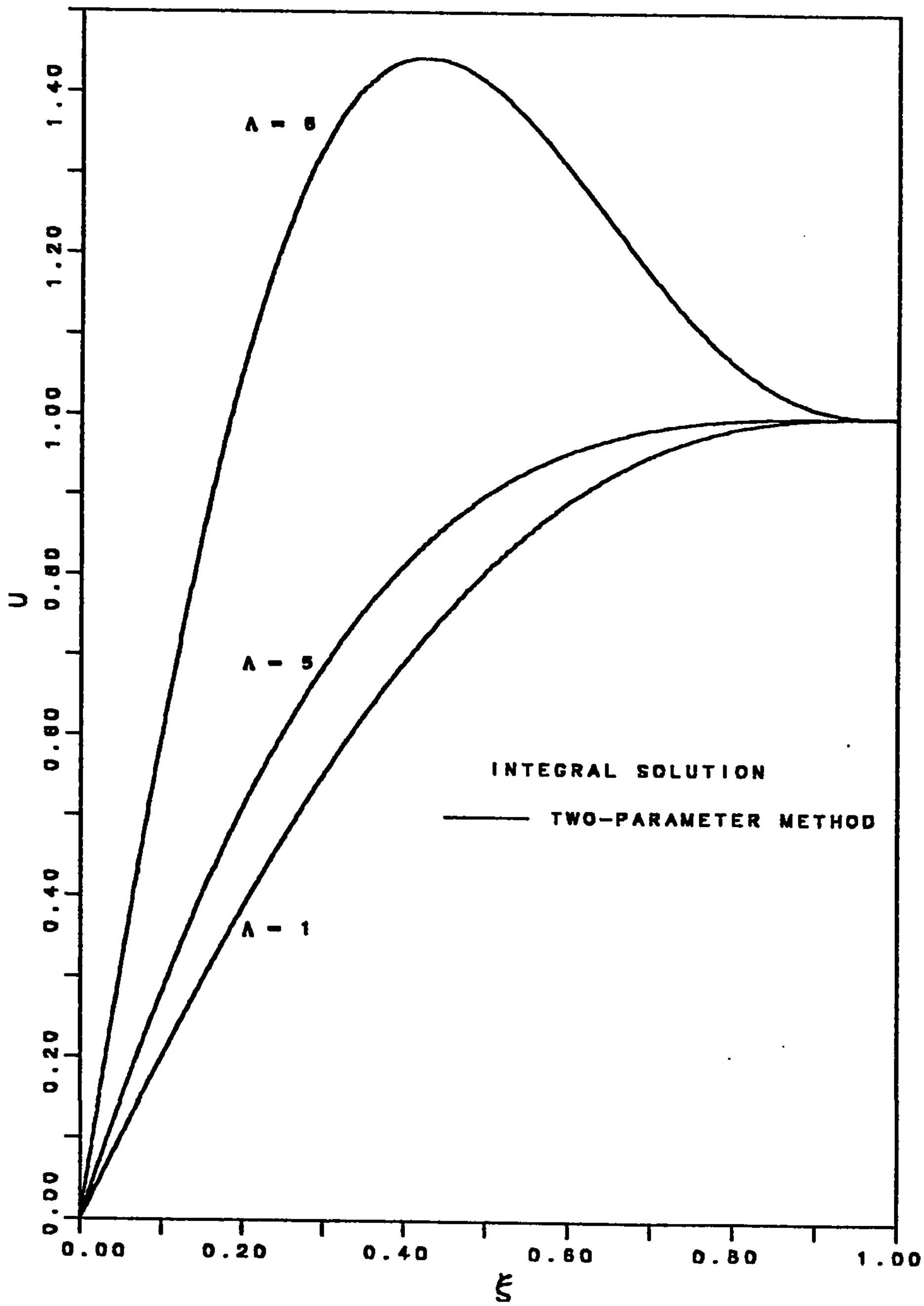

Fig. 3-1 Approximations for velocity distribution as a function of Λ.

velocity popping for mild to moderate favorable pressure gradients. Furthermore, the calculations revealed multiple roots. Therefore, it appears that two-parameter integral methods using polynomial approximation for velocity or stress cannot be used for strong favorable pressure gradient.

Other integral approaches that are feasible for favorable pressure gradients include the two-parameter method of the second kind and methods which are based on similarity profiles. However, neither of these approaches appear to provide a basis for generalization to turbulent boundary layer flow.

As a practical alternative it was decided to re-examine the one-parameter integral method of the first kind for strong favorable pressure gradients. Although one-parameter integral methods are not reliable for near separating flows, 2nd and 3rd order one-parameter integral method of the first kind have been found to provide reasonably accurate results for mild to moderate favorable pressure gradients [3]. Furthermore, this approach can be adapted to the analysis of turbulent boundary layer flow. Therefore, attention is now devoted to 4th, 5th and 6th order one-parameter approximation of the first kind.

A one-parameter 4th order polynomial approximation for stress is obtained by setting N = 4 in Eq.(2.28b),

$$\frac{\tau_{xy}}{\tau_0} = \sum_{n=0}^{4} a_n \xi^n + B_m U$$

$$= a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 + a_4\xi^4 + B_m U \qquad (3.1)$$

where relations for the four coefficients a_0, a_1, a_2, and a_3 are established by satisfying the Couette law, Eq.(2.26) and Eqs.(2.27a) and (2.27b). To provide a one-parameter closure statement by which a_4 can be evaluated, the following requirements are established

$$\frac{\partial^2 \tau_{xy}}{\partial y^2} = \text{DDTAU}, \quad \frac{\partial^2 U}{\partial y^2} = 0 \quad \text{at } \xi = 1 \qquad (3.2)$$

where DDTAU is set equal to zero. Equation (2.26), (2.27) and (3.2) are incorporated into Eq.(3.1) to obtain

$$\frac{\tau_{xy}}{\tau_0} = 1 + \beta_\delta\xi + B_m U + (-6 - 3\beta_\delta - 6B_m)\xi^2 + (8 + 3\beta_\delta + 8B_m)\xi^3$$

$$+ (-3 - \beta_\delta - 3B_m)\xi^4 \qquad (3.3)$$

The coefficients a_n for this 4th order one-parameter approximation are summarized in Table. 3-1.

Following this approach, 5th and 6th order polynomial approximations are developed for τ_{xy}/τ_0 of the form

$$\frac{\tau_{xy}}{\tau_0} = 1 + \beta_\delta\xi + B_m U - (10 + 6\beta_\delta + 10B_m)\xi^3 + (15 + 8\beta_\delta + 15B_m)\xi^4$$

$$- (6 + 3\beta_\delta + 6B_m)\xi^5 \qquad (3.4)$$

Table 3-1 Coefficients α_n, β_n and γ_n for N = 4 (One-parameter method).

α_0	α_1	α_2	α_3	α_4	β_0	β_1	β_2	β_3	β_4	γ_0	γ_1	γ_2	γ_3	γ_4
1	0	-6	8	-3	0	-1	3	-3	1	0	0	6	-8	3

and

$$\frac{\tau_{xy}}{\tau_0} = 1 + \beta_\delta\xi + B_mU - (20 + 10\beta_\delta + 20B_m)\xi^3 + (45 + 20\beta_\delta + 45B_m)\xi^4$$

$$- (36 + 15\beta_\delta + 36B_m)\xi^5 + (10 + 4\beta_\delta + 10B_m)\xi^6 \qquad (3.5)$$

respectively.

These one-parameter approximations for τ_{xy}/τ_0 have been used together with Eqs.(2.38) and (2.39) and Table. 2-3 to develop one-parameter calculations for velocity distribution and friction factors for similar nontranspired flows. The resulting calculations for the various integral parameters are listed in Tables. 3-2a, 3.2b and 3.2c. The calculations for friction factor are compared with numerical solution results in Figs. 3.2 for β ranging from -0.4 to 2. Calculations are also shown for the 3rd order Pohlhausen method. The values of Λ and β at which these one-parameter methods break down are listed in Table. 3-3. These results clearly demonstrate that the 4th order one-parameter method encompasses by far the best range of applicability for accelerating flows. Furthermore, the accuracy of the 4th order method is within 3 to 6 % over the range $0.0 \leq \beta \leq 2$. This compares to an accuracy of 3 % over the range $0.0 \leq \beta \leq 2$ for the Pohlhausen method. Thus, the 4th and 3rd order integral methods provide good accuracy and range of applicability for favorable pressure gradient flows, with the range of the 4th order method being considerably broader. However, these one-parameter methods prove

Table 3-2a One-parameter integral solution for N=4.

Λ	β	FF1	RESR
20.00	10.00372	1.24994	0.06962
19.00	9.57259	1.25745	0.06873
18.00	8.39329	1.28281	0.06951
17.00	6.91784	1.33041	0.06976
16.00	5.49313	1.41077	0.06716
15.00	4.28152	1.54746	0.06646
14.00	3.31848	1.80201	0.06399
13.00	2.57297	2.42440	0.06177
12.00	2.00015	132.39790	0.05763
11.00	1.55983	2.19939	0.05281
10.00	1.21843	1.48063	0.04707
9.00	0.95142	1.15100	0.03975
8.00	0.74036	0.94868	0.03117
7.00	0.57167	0.80702	0.02078
6.00	0.43562	0.70015	0.00866
5.00	0.32489	0.61548	-0.00532
4.00	0.23402	0.54604	-0.02133
3.00	0.15890	0.48764	-0.03964
2.00	0.09638	0.43753	-0.06037
1.00	0.04404	0.39387	-0.08354
0.00	0.00000	0.35534	-0.08354

Table 3-2b One-parameter integral solution for N=5.

Λ	β	FF1	RESR
10.00	0.65932	0.88704	0.01075
9.00	0.64651	0.87543	0.01164
8.00	0.60832	0.84149	0.01366
7.00	0.54749	0.78897	0.01599
6.00	0.46985	0.72391	0.01870
5.00	0.38277	0.65271	0.02081
4.00	0.29345	0.58054	0.02186
3.00	0.20759	0.51085	0.02107
2.00	0.12895	0.44552	0.01786
1.00	0.05955	0.38541	0.01139
0.00	0.00000	0.33066	0.01139

Table 3-2c One-parameter integral solution for N=6.

Λ	β	FF1	RESR
14.00	0.79436	1.01238	0.00579
13.00	0.78594	1.00419	0.00801
12.00	0.75944	0.97875	0.00884
11.00	0.71652	0.93867	0.00867
10.00	0.65927	0.88700	0.01057
9.00	0.59132	0.82787	0.01174
8.00	0.51649	0.76491	0.01253
7.00	0.43863	0.70116	0.01261
6.00	0.36117	0.63884	0.01198
5.00	0.28668	0.57929	0.01020
4.00	0.21700	0.52329	0.00683
3.00	0.15318	0.47111	0.00277
2.00	0.09573	0.42279	-0.00203
1.00	0.04475	0.37822	-0.01067
0.00	0.00000	0.33711	-0.01067

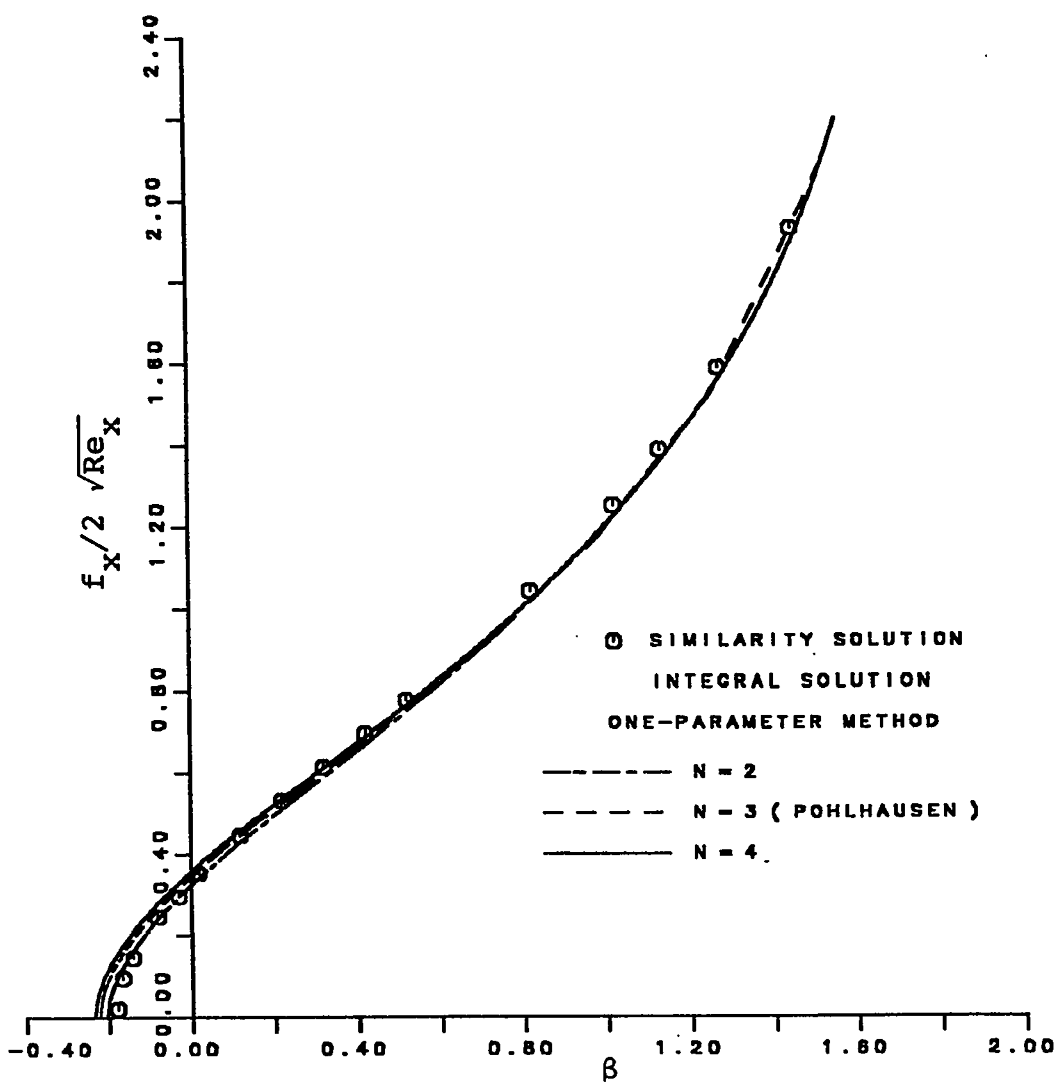

Fig. 3-2a One-parameter calculations for friction factor for similar boundary layer flow with pressure gradient for N = 2, 3, 4. ($-0.4 \leq \beta \leq 2.0$).

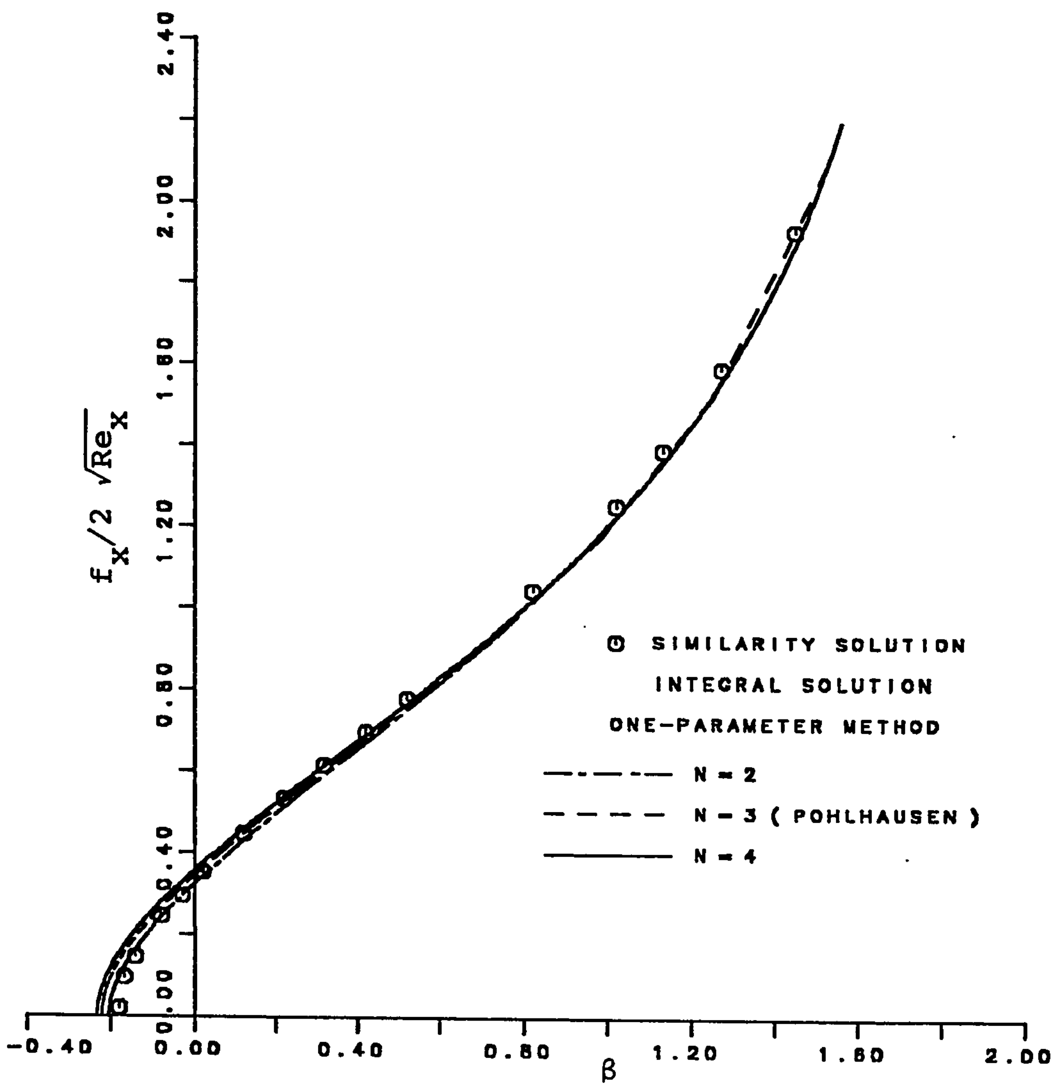

Fig. 3-2b One-parameter calculations for friction factor for similar boundary layer flow with pressure gradient for N = 5, 6. ($-0.4 \leq \beta \leq 2.0$).

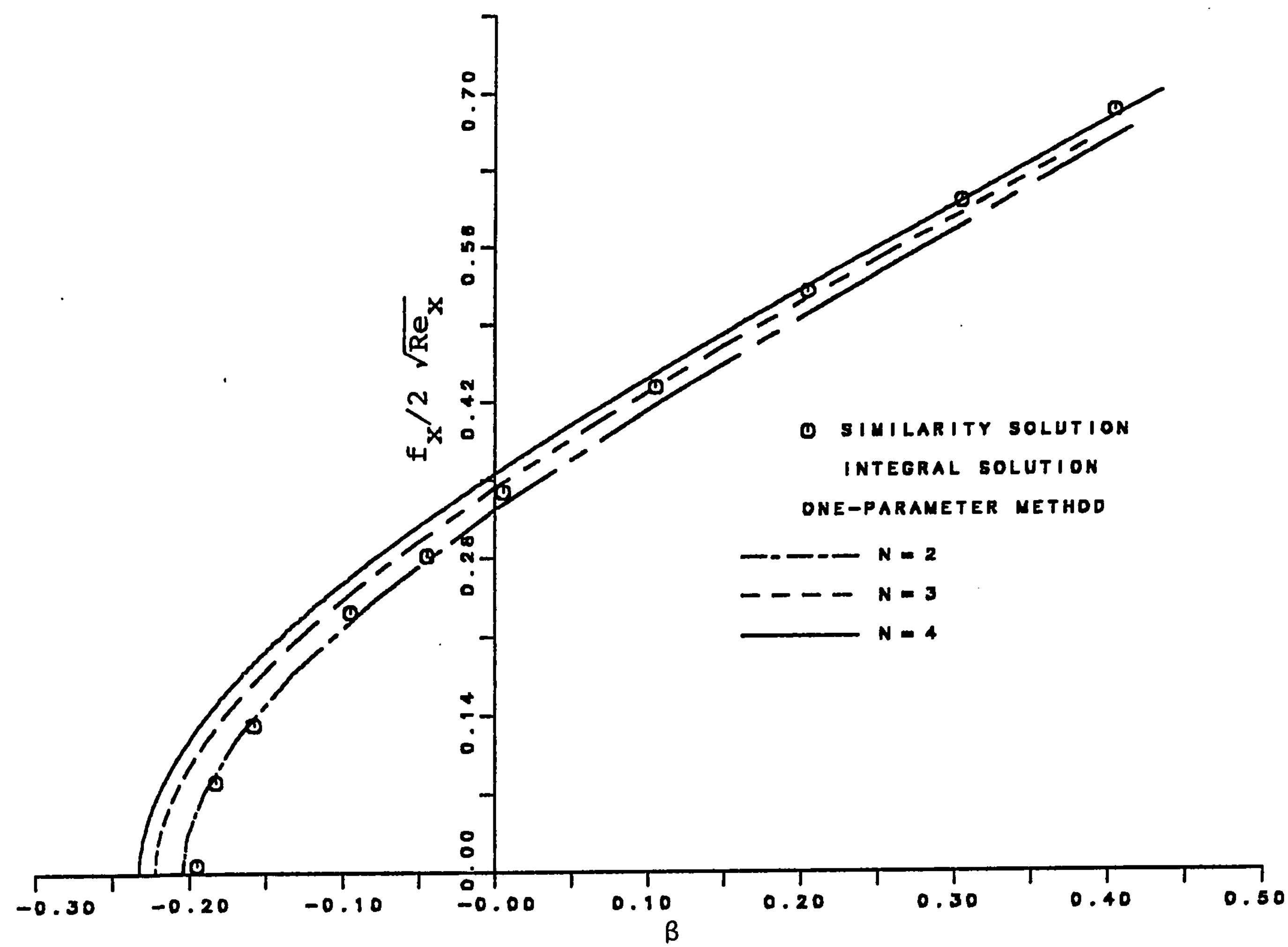

Fig. 3-2c One-parameter calculations for friction factor for similar boundary layer flow with pressure gradient for N = 2, 3, 4. ($-0.3 \le \beta \le 0.5$).

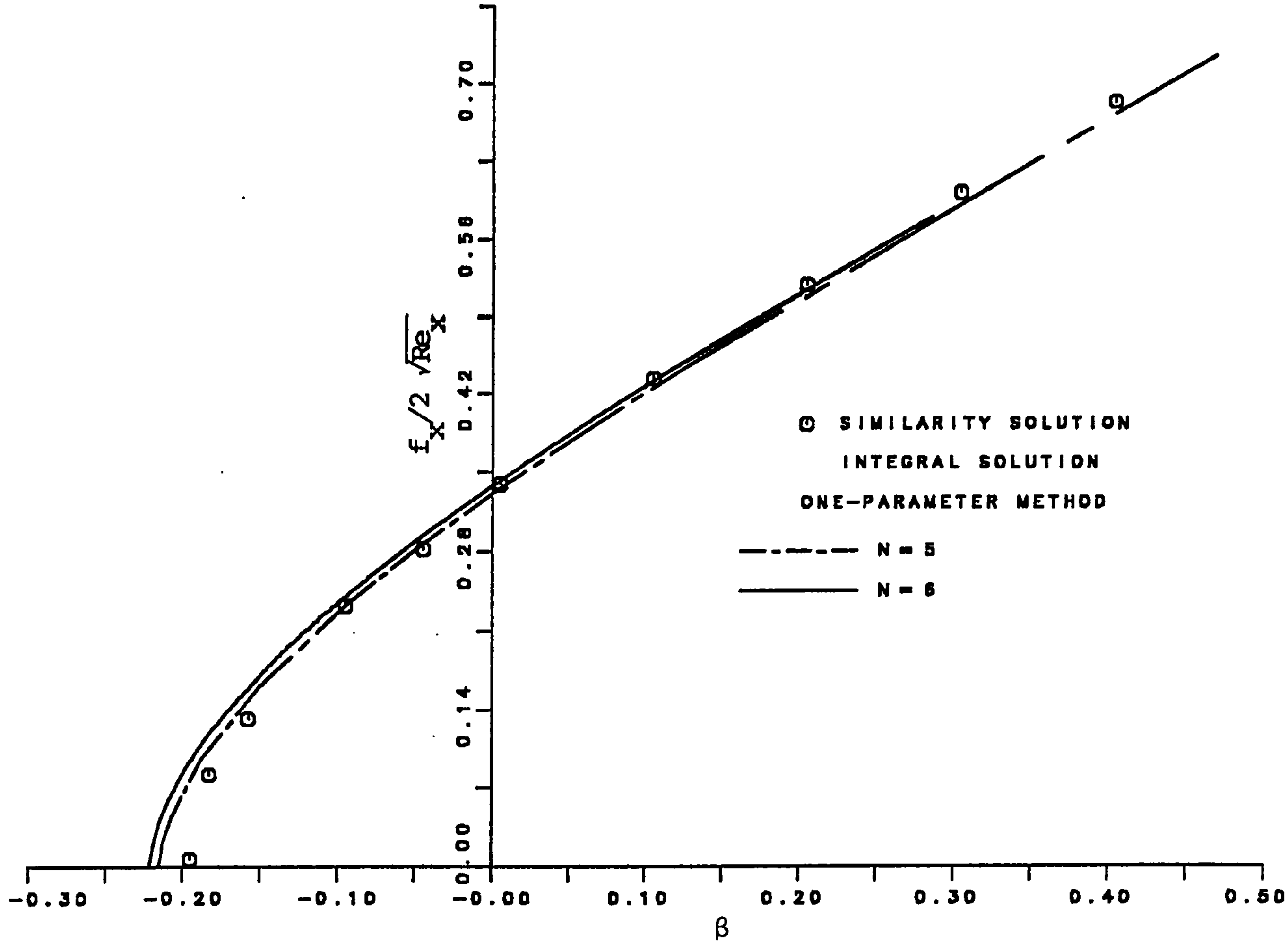

Fig. 3-2d One-parameter calculations for friction factor for similar boundary layer flow with pressure gradient for N = 5, 6. ($-0.3 \le \beta \le 0.5$).

Table 3-3 Limiting value of Λ for one-parameter integral solution.

N	Λ	β
2	6.0	0.75001
3	12.0	1.99999
4	20.0	10.00372
5	10.0	0.65932
6	14.0	0.79436

to be unreliable for adverse pressure gradient flows, especially near separation.

These results bring to light the feasibility of developing a 4th order composite integral method of the first kind which operate with one-parameter in the region of moderate to strong favorable pressure gradient and which switches over to two-parameter for near separating flows with adverse pressure gradients. The objective of this thesis is to develop a practical, elegant and reliable composite integral method of this type for transpired as well as nontranspired laminar boundary layer flow.

3.3 PROPOSED ONE AND TWO-PARAMETER COMPOSITE INTEGRAL METHOD

A 4th order composite one and two-parameter integral method is now proposed for transpired and nontranspired boundary layer flows which is based on the polynomial approximation for τ_{xy}/τ_0 given by Eq.(2.30),

$$\frac{\tau_{xy}}{\tau_0} = 1 + \beta_\delta\xi + B_mU - (3 + 2\beta_\delta + 3B_m)\xi^2 + (2 + \beta_\delta + 2B_m)\xi^3 + a_4(\xi^2 - 2\xi^3 + \xi^4) \qquad (2.30)$$

or equivalently by

$$\frac{\tau_{xy}}{\tau_0} = 1 + \beta_\delta\xi + B_mU - (6 + 3\beta_\delta + 6B_m + \frac{DDTAU}{2})\xi^2 + (8 + 3\beta_\delta + 8B_m - DDTAU)\xi^3 -$$

$$\left(- \frac{DDTAU}{2} + 3 + \beta_\delta + 3B_m \right)\xi^4 \tag{3.6}$$

where a_4 and DDTAU are related by

$$DDTAU = 2a_4 + 2\beta_\delta + 6B_m + 6 \tag{3.7}$$

$$a_4 = \frac{DDTAU}{2} - \beta_\delta - 3B_m - 3 \tag{3.8}$$

The method operates in the noniterative one-parameter mode simply by setting DDTAU equal to zero and in the two-parameter mode by evaluating a_4 on the basis of the integral mechanical energy equation. In this connection, it has been found expedient to express a_4 by

$$a_4 = \alpha_4 - \beta_4\beta_\delta - \gamma_4 B_m \tag{3.9}$$

with α_4 (or β_4, or γ_4) taken as the dependent variable and the other two coefficients (β_4 and γ_4 in this case) treated as constants.

Because of the nature of the integral mechanical energy equation and the integral relations, a_4 must be generally evaluated by iterative methods. In the present analysis, the Newton-Raphson technique is used to iterate on α_4. In this method, the relative error or residue R,

$$R = \varphi(\alpha_4) - \varphi_k(\alpha_{4_k}) \tag{3.10}$$

is approximated by the first order Taylor series expansion

$$R = R_k + \frac{\partial R}{\partial \alpha_4}\Big|_k \Delta\alpha_4 \tag{3.11}$$

where φ is the exact solution, φ_k is the kth iterative solution, and α_4 is the primary unknown parameter. Equation (3.11) is used to establish the k+1 th iterative value $\alpha_{4_{k+1}}$ by setting R = 0 and writing

$$\alpha_{4_{k+1}} = \alpha_{4_k} + \Delta\alpha_4 \tag{3.12}$$

where

$$\Delta\alpha_4 = -\frac{R_k}{\left(\frac{\partial R}{\partial \alpha_4}\right)_k} = -\frac{R_k}{(R_k - R_{k-1})/(\alpha_{4_k} - \alpha_{4_{k-1}})}$$

With $\varphi_1(\alpha_{4_1})$ and $\varphi_2(\alpha_{4_2})$ specified, R_1 and R_2 can be determined, after which φ_k and R_k can be calculated for k = 3, 4,........ . Assuming that the residue R_k is a monotonous function of α_{4_k}, the solution is terminated when the desired accuracy is obtained.

In the composite integral method developed in this thesis, the integral momentum equation is used to obtain explicit finite difference calculations for Λ and Ω at the next station. a_4 is calculated by setting DDTAU = 0 in the one-parameter mode or by satisfying the integral mechanical energy equation in the two-parameter iterative mode. The integral parameters Mo_δ, δ_1/δ, δ_2/δ, δ_3/δ, D_1 are expressed in terms of Λ, Ω and a_4 given by relations shown in Table. 2-2. The integral parameters are calculated by the use of nested do loops. Once these integral parameters are known, all the other dependent variables can be computed by using Eqs.(2.39), and the calculations can be extended to the following station. The method stops at the location at which separation occurs.

For boundary layer flow with moderate to strong acceleration and suction at the first station, the method is started and operated in the one-parameter mode with the residue computed by the use of Eq.(2.47) for reference. The method is switched to the two-parameter mode when the residue changes sign. For boundary layer with adverse pressure gradient and blowing or mild favorable pressure gradient and suction at the first station, the method is started and operated in the two-parameter mode. The method is switched from two-parameter to one-parameter operation when the residue is not satisfied.

For nontranspired flow $v_0 = 0$, $\Omega = 0$ and $B_m = 0$, such that the integral parameters and the dependent variables are expressed in terms of Λ and a_4 only. The integral relations for nontranspired flow are listed in Table. 2-3.

3.4 RESULTS

Results obtained for friction factors and velocity distributions by use of the proposed composite integral method are presented in this section for both transpired and nontranspired flows.

3.4.1 Nontranspired Flows

For nontranspired flows $\Omega = 0$ and the integral parameters and flow characteristics are calculated by using the equations listed in Table. 2-3. Solution results are presented for both similar and nonsimilar flows.

3.4.1.1 Similar Flows

Solution results for the integral parameters and flow characteristics obtained by the composite one and two-parameter integral method developed in this study are presented in Table. 3-4 as a function of Λ for standard Falkner-Skan wedge flows. The value of Λ ranges from -5.23098 to 20. This table indicates one-parameter operation (DDTAU = 0) for $6 \leq \Lambda \leq 20$, and two-parameter operation (DDTAU $\neq$ 0) for $-5.23098 \leq \Lambda \leq 5$.

The dimensionless friction factor, $(f_x/2)\sqrt{Re_x}$ is plotted against β in Fig. 3-3. Numerical solution results are also shown for comparison. The accuracy of the composite solution ranges from about 1% from separation to moderate favorable pressure gradients and 3 to 4% for moderate to strong favorable pressure gradients.

As can be seen from the Table. 3-4, the variation in α_4 in the two-parameter zone is much smaller compared with the variation of a_4. Thus the use of α_4 generally leads to quicker convergence when

Table 3-4a Distribution in integral parameters as a function of Λ for $\Omega=0$.

Λ	δ_2/δ	S	H	λ	D_1	F_2
20.00	0.07576	0.37880	2.19994	0.11479	0.21045	-0.20663
19.00	0.07763	0.37847	2.20049	0.11452	0.21034	-0.20511
18.00	0.07942	0.37727	2.20336	0.11355	0.20991	-0.20004
17.00	0.08116	0.37536	2.20761	0.11198	0.20926	-0.19158
16.00	0.08284	0.37277	2.21319	0.10979	0.20840	-0.17961
15.00	0.08444	0.36944	2.22045	0.10696	0.20734	-0.16395
14.00	0.08599	0.36546	2.22893	0.10352	0.20610	-0.14465
13.00	0.08748	0.36084	2.23872	0.09948	0.20472	-0.12163
12.00	0.08889	0.35556	2.24998	0.09482	0.20318	-0.09482
11.00	0.09025	0.34971	2.26229	0.08959	0.20154	-0.06431
10.00	0.09154	0.34327	2.27591	0.08379	0.19979	-0.03004
9.00	0.09277	0.33629	2.29061	0.07746	0.19797	0.00791
8.00	0.09394	0.32880	2.30635	0.07060	0.19609	0.04952
7.00	0.09505	0.32078	2.32340	0.06324	0.19415	0.09476
6.00	0.09609	0.31229	2.34160	0.05540	0.19218	0.14354
5.00	0.09977	0.30590	2.35607	0.04977	0.19066	0.17817
4.00	0.10609	0.29808	2.37881	0.04502	0.18829	0.20189
3.00	0.11117	0.28490	2.41728	0.03708	0.18464	0.24223
2.00	0.11555	0.26714	2.47059	0.02670	0.18018	0.29554
1.00	0.11924	0.24495	2.54031	0.01422	0.17524	0.36079
0.00	0.12226	0.21803	2.63100	0.00000	0.17009	0.43607
-1.00	0.12442	0.18615	2.74920	-0.01548	0.16506	0.51933
-2.00	0.12544	0.14927	2.90450	-0.03147	0.16060	0.60722
-3.00	0.12492	0.10732	3.11463	-0.04682	0.15714	0.69356
-4.00	0.12228	0.06090	3.40911	-0.05981	0.15510	0.76882
-5.00	0.11654	0.01124	3.85019	-0.06790	0.15454	0.81699
-5.23098	0.11471	0.00007	3.97825	-0.06883	0.15455	0.82313

Table 3-4b Distribution in integral parameters as a function of Λ for $\Omega=0$.

Λ	β	FF1	a_4	α_4	RESR	DDTAU
20.00	10.00372	1.24994	1.00000	13.80001	0.06962	0.0000
19.00	9.57259	1.25745	0.89744	13.36924	0.06873	0.0000
18.00	8.39329	1.28281	0.78948	12.91580	0.06951	0.0000
17.00	6.91784	1.33041	0.67568	12.43784	0.06976	0.0000
16.00	5.49313	1.41077	0.55556	11.93334	0.06716	0.0000
15.00	4.28152	1.54746	0.42857	11.40000	0.06646	0.0000
14.00	3.31848	1.80201	0.29412	10.83530	0.06399	0.0000
13.00	2.57297	2.42440	0.15152	10.23637	0.06177	0.0000
12.00	2.00015	132.39790	0.00000	9.60000	0.05763	0.0000
11.00	1.55983	2.19939	-0.16129	8.92258	0.05281	0.0000
10.00	1.21843	1.48063	-0.33333	8.20000	0.04707	0.0000
9.00	0.95142	1.15100	-0.51724	7.42759	0.03975	0.0000
8.00	0.74036	0.94868	-0.71428	6.60000	0.03117	0.0000
7.00	0.57167	0.80702	-0.92593	5.71111	0.02078	0.0000
6.00	0.43562	0.70015	-1.15385	4.75385	0.00866	0.0000
5.00	0.35844	0.64071	-1.15401	4.06458	0.00028	0.4303
4.00	0.30843	0.59987	-0.78550	3.77017	0.00056	1.5817
3.00	0.23438	0.53907	-0.38618	3.35991	0.00058	2.8863
2.00	0.15304	0.47060	0.11816	2.88631	0.00067	4.5062
1.00	0.07306	0.40000	0.81180	2.36953	0.00030	6.6389
0.00	0.00000	0.33018	1.81948	1.81948	0.00008	9.6389
-1.00	-0.06339	0.26225	3.38167	1.24286	-0.00004	14.100
-2.00	-0.11563	0.19673	6.02146	0.64347	-0.00039	21.4041
-3.00	-0.15608	0.13345	11.20597	0.03124	-0.00034	35.3961
-4.00	-0.18425	0.07233	25.15392	-0.54567	0.00002	72.3700
-5.00	-0.19937	0.01298	166.14630	0.22918	0.00086	441.9907
-5.23	-0.20083	0.00008	*********	632.39740	-0.00078	********

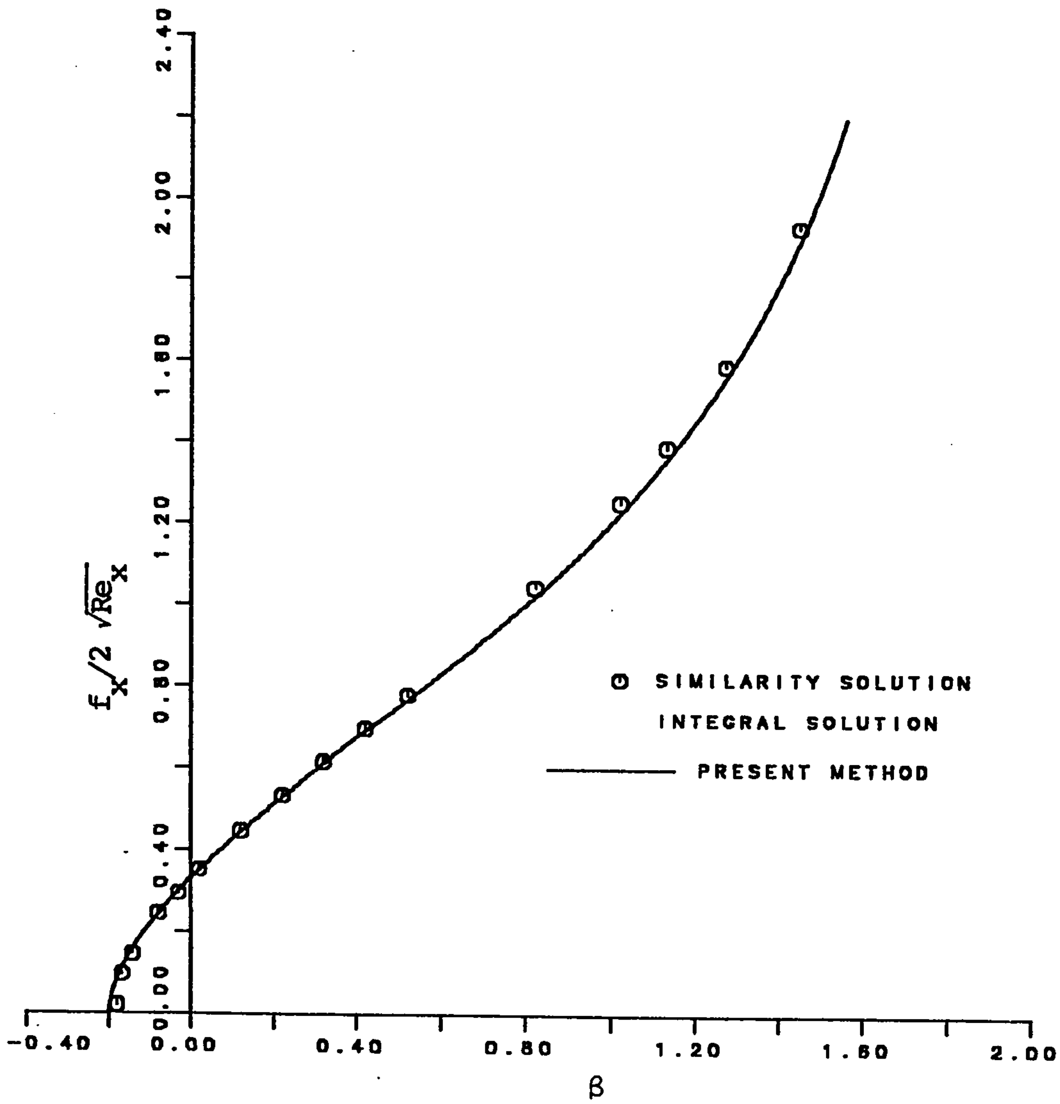

Fig. 3-3 Calculations for friction factor for similar boundary layer flow with pressure gradient.

iterating. The number of iterations taken at each step is lower, this is indeed evident from the Table. 3-4. Hence the iterative scheme is very efficient and requires less computer time.

Solution results for friction factor associated with large favorable pressure gradients are generally expressed in the format $(f_x/2)/\sqrt{K}$. In the light of this perspective the integral solution for friction factor is given by

$$\frac{f_x/2}{\sqrt{K}} = \frac{S}{\sqrt{\lambda}} \tag{3.13}$$

where K is the acceleration parameter ($\equiv \nu/U_\infty^2 dP/dx$). $(f_x/2)/\sqrt{K}$ is plotted against large positive values of β in Fig. 3-4 and compared with exact similarity solution.

Integral solution results for velocity distributions U are compared with numerical calculations in Fig. 3-5 for β = 1, 0.5, 0, -0.18 and separation. There is a close matching between the composite integral and numerical solutions for U.

3.4.1.2 Nonsimilar Flows

To test the composite integral method developed for nonsimilar nontranspired boundary layer flow, attention is confined to the following flows

1. Linear retarded flow
2. Linear accelerated flow

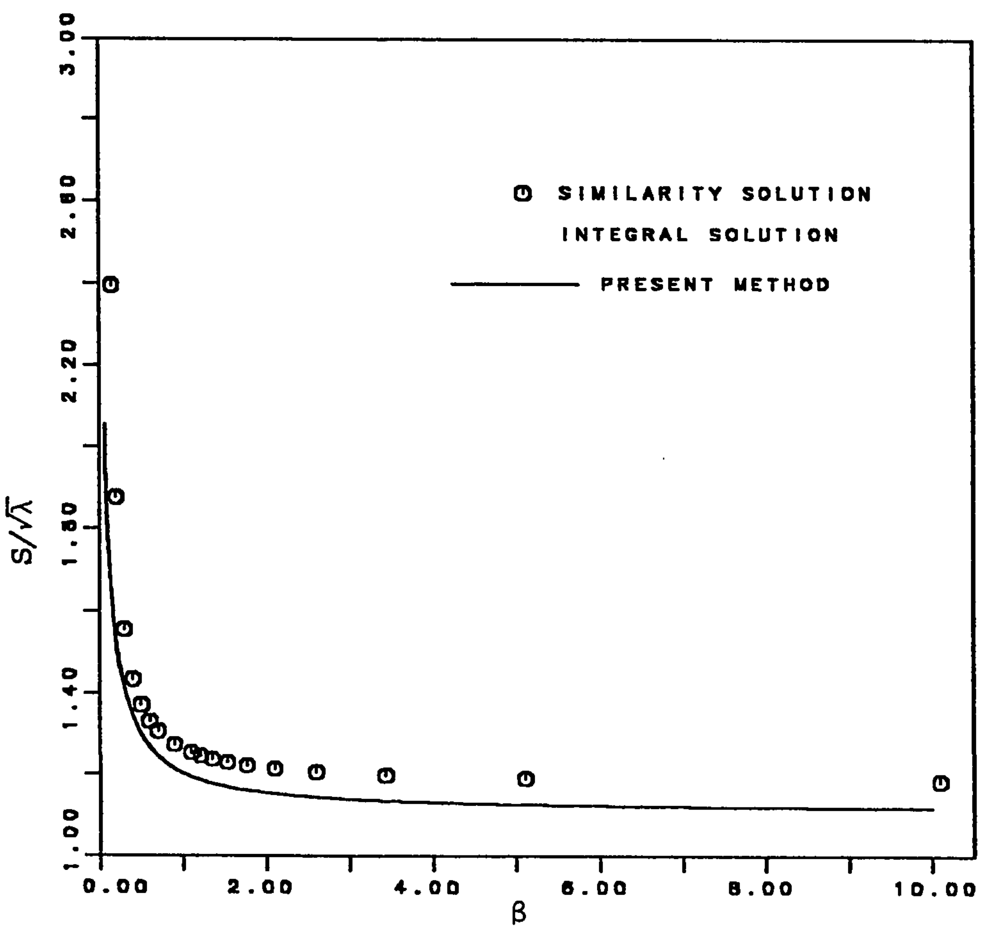

Fig. 3-4 Calculations for friction factor for similar boundary layer flow with large favorable pressure gradient.

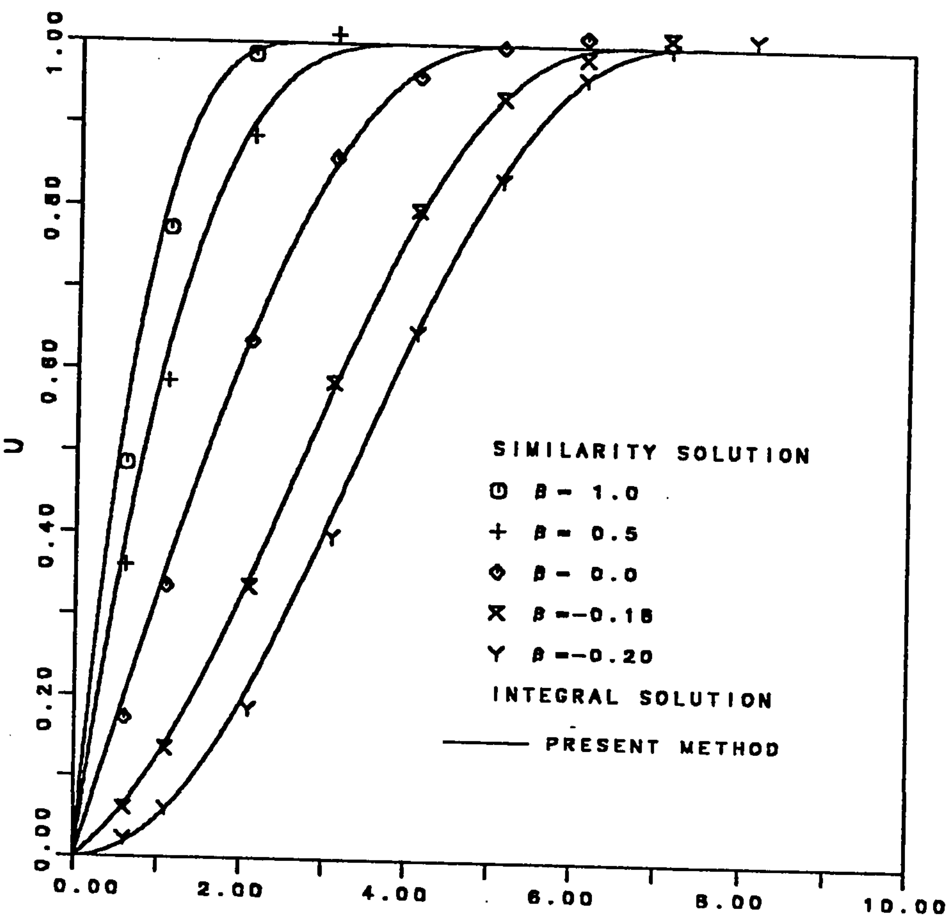

Fig. 3-5 Comparison of integral solution for velocity distribution with similarity solution for nontranspired boundary layer flow.

3. Plane flow over a circular cylinder
4. Axisymmetric flow over a sphere

3.4.1.2.1 Linear Retarded Flow

For the linear retarded flow of Howarth [16], the free stream velocity is given by

$$U_\infty = u_0(1 - 0.125\frac{x}{L}) \tag{3.14}$$

where u_0 and L are constants.

For this flow the method was initiated by setting $\Lambda = 0$ and $U_\infty\delta/\nu = 0$ at the first station, $x/L = 0$. An increment of $\Delta x = 0.01$ was found to provide sufficient accuracy. Because this is an adverse pressure gradient flow over the entire length, the method can operate solely in the two parameter mode.

The friction factor obtained by the use of composite integral method is compared with numerical calculations given by Cebeci and Smith [17] in Fig. 3-6. The difference between the integral solution and the numerical solution is less than 1% except near the separation point where the error in x/L is about 1.6%. Results obtained by using 2nd, 3rd, and 4th order one-parameter modes are also shown in fig. 3-6. The error at separation associated with these approaches are 11% for N = 2, 24% for N = 3, and 30% for N = 4. These results demonstrate the usefulness of the two parameter approach for adverse pressure gradient flow.

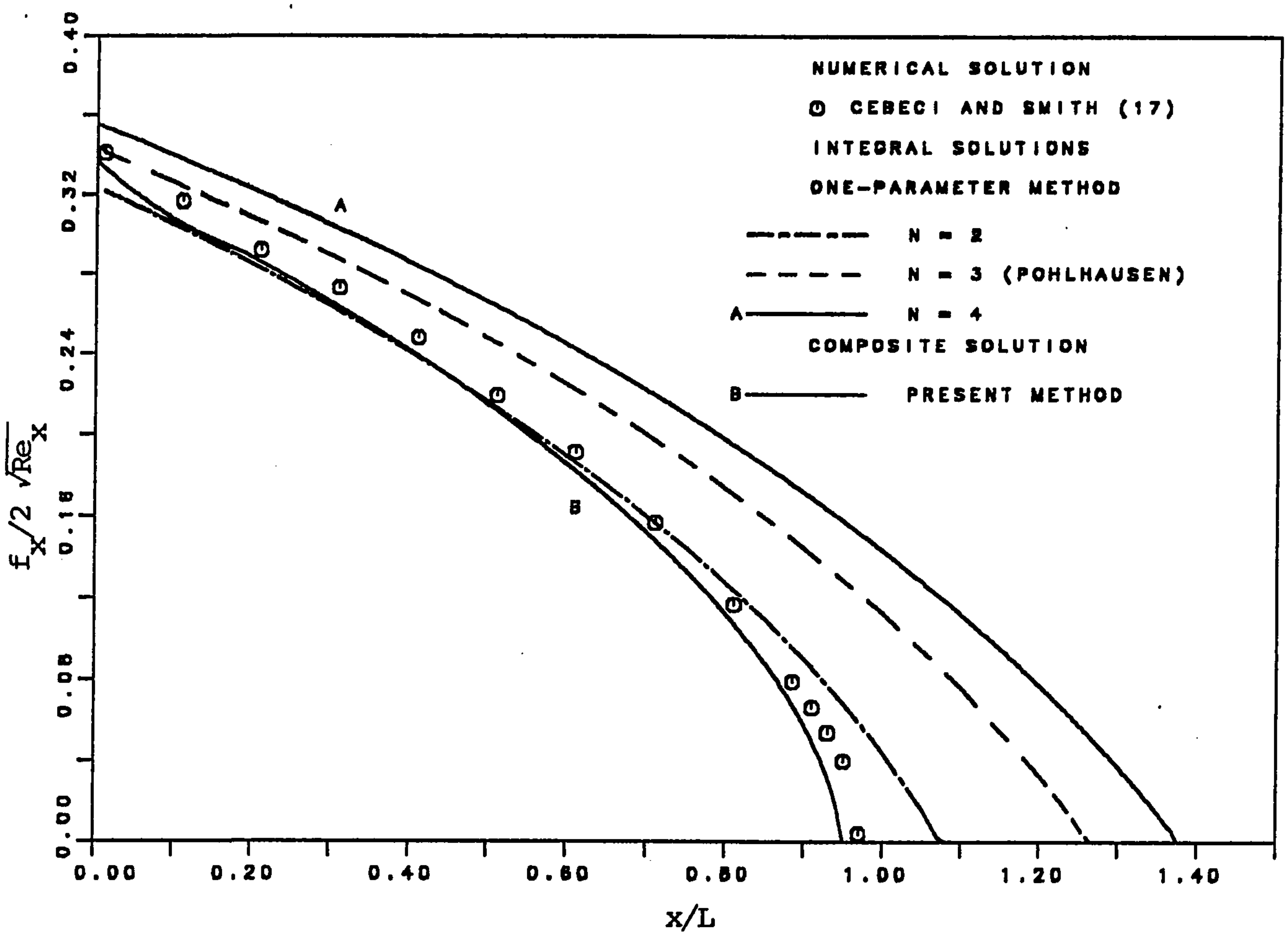

Fig. 3-6 Calculations for friction factor for nonsimilar linear retarded flow.

3.4.1.2.2 Linear Accelerated Flow

For linear accelerated flow, the free stream velocity is given by

$$U_\infty = u_0(1 + 0.125\frac{x}{L}) \tag{3.15}$$

where u_0 and L are constants.

For this flow the method is started by setting $\Lambda = 0$ and $U_\infty\delta/\nu = 0$ at the first station, $x/L = 0$, with Δx set equal to 0.01. To start with the method is executed at the first station and operated as a two-parameter integral method for mild acceleration. When the residue criterion fails to be satisfied, the method switches over and operates as a one-parameter method. The friction factor obtained by the composite integral method is plotted against x/L. in Fig. 3-7. For $0 \leq \Lambda \leq 6$ the method functions as a two-parameter method and for $\Lambda > 6$ the method operates as a one-parameter method.

3.4.1.2.3 Plane Flow Over a Circular Cylinder

For plane flow over a circular cylinder the free stream velocity is represented by a correlation developed by Hiemenz [18]

$$\frac{U_\infty}{u_0} = 1.814(\frac{x}{R_0}) - 0.2710(\frac{x}{R_0})^3 - 0.04710(\frac{x}{R_0})^5 \tag{3.16}$$

for actual conditions, and by

$$\frac{U_\infty}{u_0} = 2\sin(\frac{x}{R_0}) \tag{3.17}$$

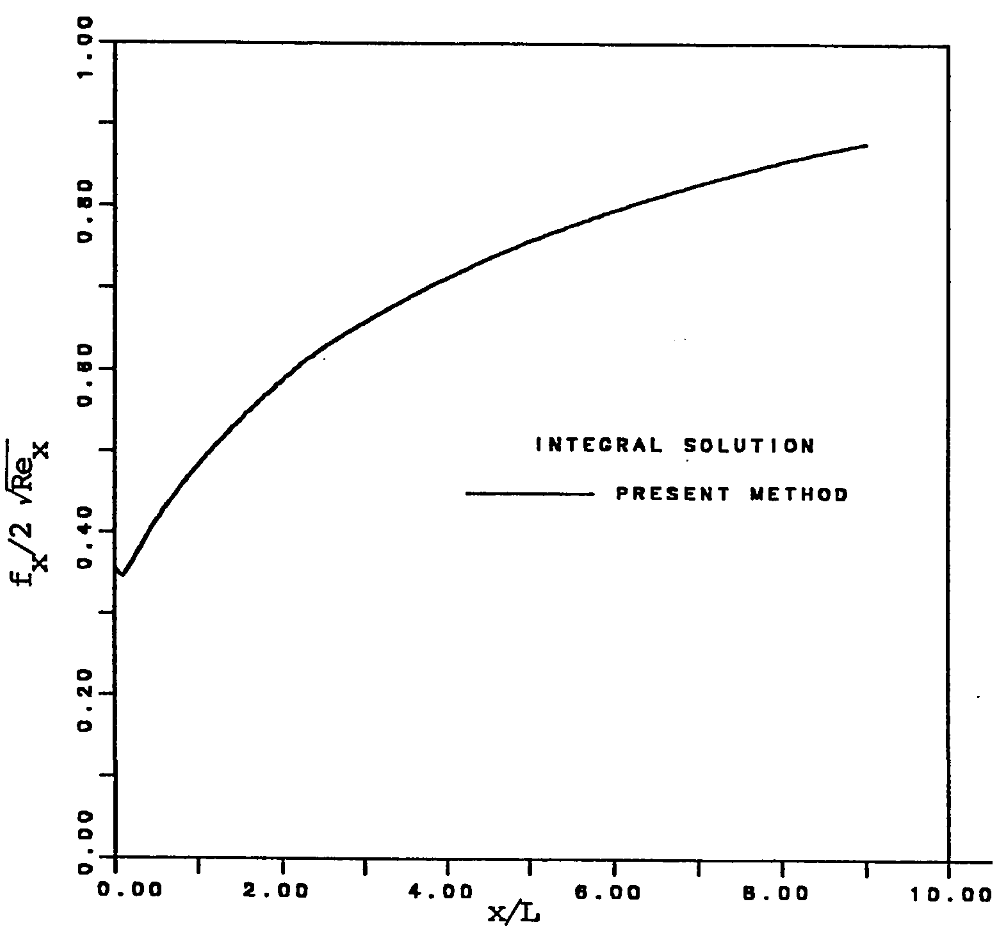

Fig. 3-7 Calculations for friction factor for nonsimilar linear accelerated flow.

for ideal potential flow, where R_0 is the cylinder radius and x is the arc length measured from the stagnation point.

Integral and numerical calculations for friction factor accomplished for the actual and potential flows are compared in Fig. 3-8. The Pohlhausen solution obtained by using the one-parameter integral method with N = 3 and the composite one-parameter solution are also shown. The composite one-parameter solution is acheived by using N = 3 for moderate acceleration and N = 2 for mild acceleration and deacceleration. The difference between the composite one and two-parameter integral solution and numerical solution is generally within 1% except near the separation where the calculations for x/L are within about 1.6%. The accuracy of composite one-parameter solution is within about 3% from stagnation to separation as compared to 3 to 17% for the Pohlhausen method.

3.4.1.2.4 Axisymmetric Flow Over a Sphere

For axisymmetric flow over a sphere, the free stream velocity can be represented by a correlation developed by Fage [19],

$$\frac{U_\infty}{u_0} = 1.5\left(\frac{x}{R_0}\right) - 0.4371\left(\frac{x}{R_0}\right)^3 + 0.14810\left(\frac{x}{R_0}\right)^5 - 0.0423\left(\frac{x}{R_0}\right)^7 \qquad (3.18)$$

for actual flow, and

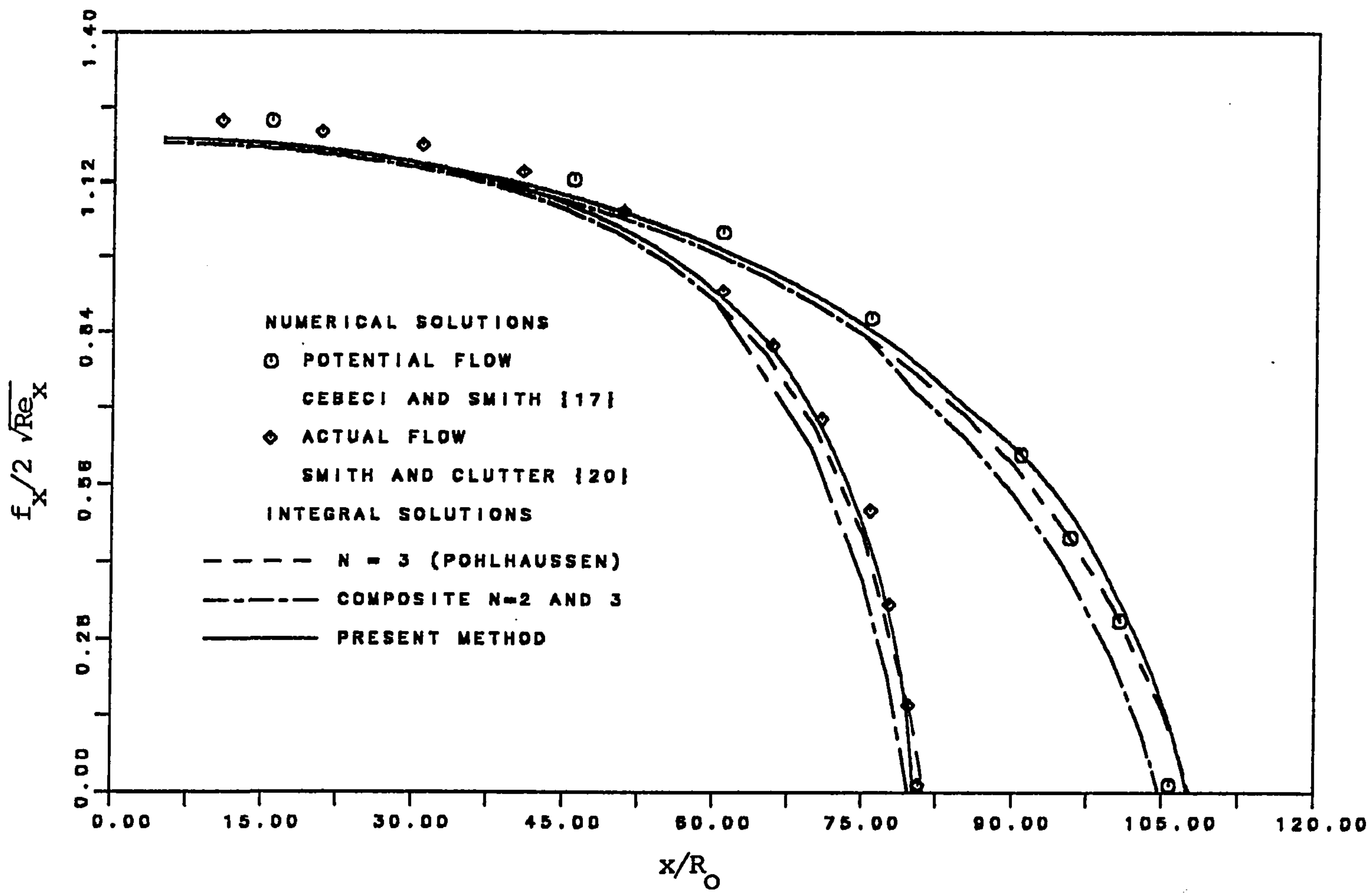

Fig. 3-8 Calculations for friction factor for nonsimilar boundary layer flow over a circular cylinder.

$$\frac{U_\infty}{u_0} = 1.5 \sin(\frac{x}{R_0}) \tag{3.19}$$

for potential flow. The local transverse radius of curvature r_0 is given by

$$r_0 = R_0 \sin(\frac{x}{R_0}) \tag{3.20}$$

Integral and numerical calculations for friction factor obtained for the actual and potential flows are compared in Fig. 3-9. The Pohlhausen solution (N = 3) and the composite one-parameter solution are also shown. The accuracy of the integral solution results for flow over a sphere is comparable to the accuracy achieved for flow over a circular cylinder.

3.4.2 Transpired Flows

For transpired flows $\Omega \neq 0$ and the integral parameters and flow characteristics are calculated by using the equations cited in Table. 2-2. Solution results for both similar and nonsimilar flows are presented in this section.

3.4.2.1 Similar Flows

3.4.2.1.1 Uniform Free Stream Velocity Flow

For uniform free stream velocity flow β_δ, Λ, λ are equal to zero. Calculations for the various integral parameters for similar uniform free stream velocity flow with blowing and suction are listed in Table. 3-5.

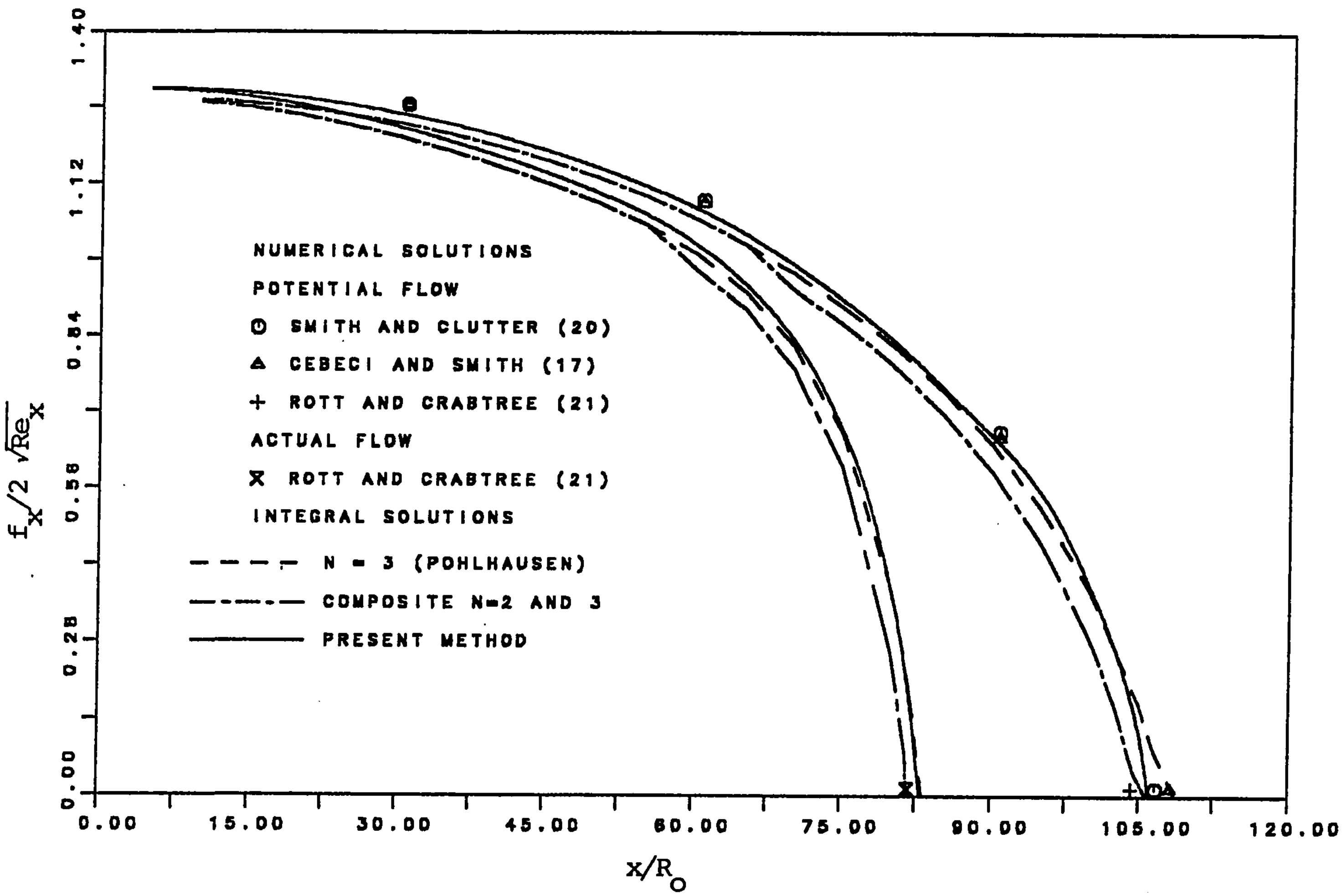

Fig. 3-9 Calculations for friction factor for nonsimilar boundary layer flow over a sphere.

Table 3-5a Distribution in integral parameters as a function of Ω for $\Lambda=0$.

Ω	δ_2/δ	S	H	D_1	Ω_2	F_2
20.00	0.02500	0.50000	2.00001	0.25000	0.49999	0.00000
19.00	0.02632	0.49999	2.00002	0.25000	0.49999	0.00000
18.00	0.02778	0.49998	2.00004	0.24999	0.49997	0.00000
17.00	0.02941	0.49995	2.00009	0.24998	0.49995	0.00001
16.00	0.03124	0.49989	2.00020	0.24995	0.49988	0.00002
15.00	0.03332	0.49976	2.00044	0.24989	0.49973	0.00005
14.00	0.03567	0.49950	2.00089	0.24978	0.49944	0.00012
13.00	0.03837	0.49899	2.00178	0.24955	0.49885	0.00027
12.00	0.04148	0.49802	2.00346	0.24914	0.49772	0.00059
11.00	0.04505	0.49621	2.00654	0.24838	0.49557	0.00126
10.00	0.04916	0.49298	2.01196	0.24705	0.49164	0.00266
9.00	0.05386	0.48746	2.02108	0.24485	0.48473	0.00547
8.00	0.05914	0.47855	2.03570	0.24143	0.47309	0.01092
7.00	0.06493	0.46500	2.05786	0.23646	0.45449	0.02101
6.00	0.07107	0.44578	2.08955	0.22981	0.42642	0.03872
5.00	0.07731	0.42054	2.13217	0.22165	0.38656	0.06795
4.00	0.08343	0.39008	2.18571	0.21256	0.33371	0.11273
3.00	0.09706	0.36423	2.23853	0.20465	0.29119	0.14608
2.00	0.10719	0.32140	2.33349	0.19284	0.21437	0.21405
1.00	0.11574	0.27112	2.46324	0.18072	0.11574	0.31076
-1.00	0.12632	0.16639	2.84145	0.16212	-0.12632	0.58543
-2.00	0.12779	0.11954	3.10010	0.15723	-0.25557	0.75023
-3.00	0.12665	0.07931	3.41648	0.15507	-0.37995	0.91853
-4.00	0.12298	0.04599	3.81098	0.15479	-0.49193	1.07582
-5.00	0.11673	0.01922	4.32115	0.15525	-0.58364	1.20573
-5.889	0.10849	0.00000	4.95279	0.15502	-0.63889	1.27778

Table 3-5b Distribution in integral parameters as a function of Ω for $\Lambda=0$.

Ω	BP	FF1	a_4	α_4	RESR	DDTAU	ITERS
20.00	-572.43033	572.43077	-0.00000	3.20000	0.00000	0.0000	0
19.00	-394.51830	394.51893	-0.00000	3.19999	0.00000	0.0000	0
18.00	-242.85651	242.85754	-0.00001	3.19997	0.00000	0.0000	0
17.00	-166.88284	166.88434	-0.00003	3.19994	0.00000	0.0000	0
16.00	-104.49856	104.50095	-0.00007	3.19986	0.00001	0.0000	0
15.00	-68.35976	68.36341	-0.00016	3.19967	0.00001	0.0000	0
14.00	-45.41987	45.42537	-0.00036	3.19925	0.00003	0.0000	0
13.00	-30.43347	30.44167	-0.00081	3.19833	0.00006	0.0000	0
12.00	-20.53901	20.55112	-0.00177	3.19634	0.00013	0.0000	0
11.00	-13.94130	13.95907	-0.00382	3.19211	0.00026	0.0000	0
10.00	-9.52722	9.55303	-0.00810	3.18325	0.00049	0.0000	0
9.00	-6.55105	6.58805	-0.01685	3.16518	0.00089	0.0000	0
8.00	-4.52685	4.57910	-0.03423	3.12925	0.00150	0.0000	0
7.00	-3.13545	3.20793	-0.06778	3.05992	0.00228	0.0000	0
6.00	-2.16696	2.26535	-0.13030	2.93071	0.00293	0.0000	0
5.00	-1.48291	1.61324	-0.24238	2.69908	0.00257	0.0000	0
4.00	-0.99392	1.16180	-0.32997	2.40764	-0.00060	0.2070	1
3.00	-0.76187	0.95297	-0.23089	2.32740	0.00013	0.7414	2
2.00	-0.46335	0.69468	-0.00082	2.13358	0.00018	1.9963	3
1.00	-0.20763	0.48635	0.55350	1.91960	0.00020	4.5455	3
-1.00	0.16510	0.21747	4.56215	2.13285	0.00002	19.6792	4
-2.00	0.29507	0.13801	10.41559	3.57416	-0.00015	39.6588	4
-3.00	0.39644	0.08276	23.27422	7.94456	-0.00056	81.2915	4
-4.00	0.47427	0.04434	55.21451	20.98330	-0.00023	180.6125	5
-5.00	0.53152	0.01751	********	74.28083	-0.00038	531.0194	6
-5.889	0.56519	0.00000	********	********	-0.00097	********	15

The value of Ω ranges from 20 to -5.889, at which separation occurs. This table indicates one parameter operation (DDTAU = 0) for strong suction with $5 \leq \Omega \leq 20$ and two parameter operation (DDTAU $\neq$ 0) for mild suction to blowing with $-5.889 \leq \Omega \leq 4$.

Integral and numerical solutions for dimensionless friction factor $(f_x/2)\sqrt{Re_x}$ are plotted against BP in Fig. 3-10. The accuracy of the composite solution varies from about 1% for mild suction to 7% for blowing near separation.

As discussed earlier for nontranspired similar flows, the variation in α_4 in the two-parameter zone is much smaller compared with the variation of a_4. Hence the convergence of solution is much quicker and the number of iterations taken at each step is also lower. This clearly reflects that the iterative scheme is very efficient and requires less time for transpired similar flows.

Solution results for the velocity distribution U are compared with numerical calculations in Fig. 3-11 for BP = -0.5, -0.25, 0.25 and 0.5. The accuracy of the integral solution is generally within 1%.

3.4.2.1.2 Plane Stagnation Flow

Plane stagnation flow is characterized by m = 1 and $F_2 = 0$. Calculations for the integral parameters adjoined with transpired plane stagnation flow are listed in Table. 3-6. The dimensionless friction

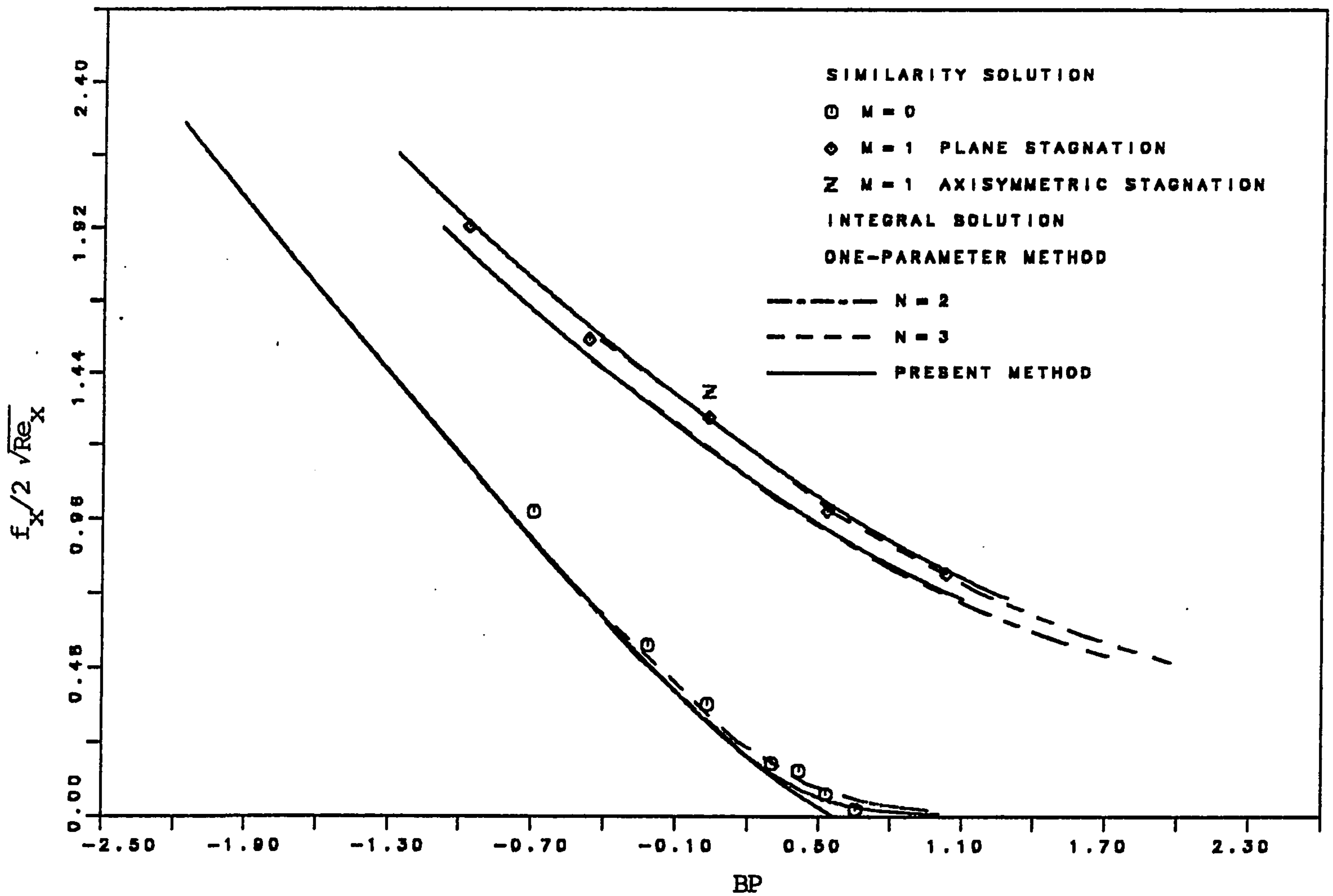

Fig. 3-10 Calculations for friction factor for similar transpired boundary layer flow.

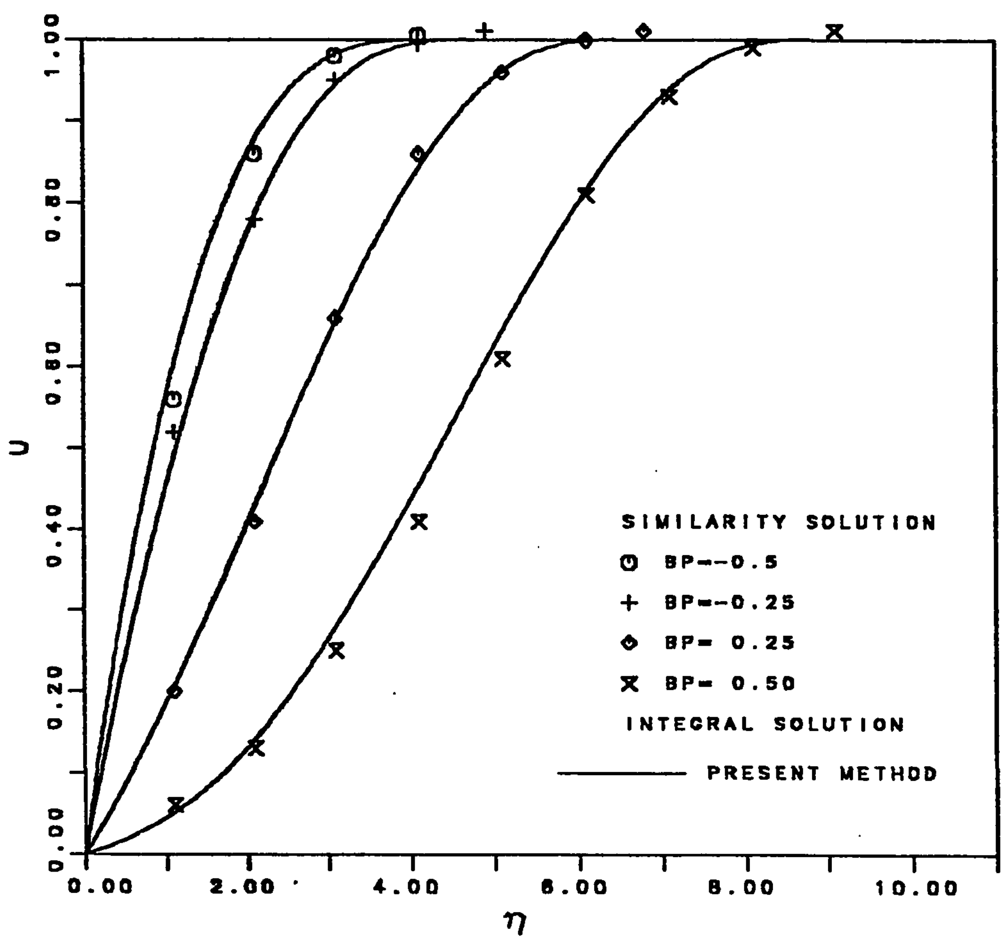

Fig. 3-11 Velocity distributions for similar flow with transpiration and uniform free stream velocity flow.

Table 3-6a Distribution in integral parameters for plane stagnation flow.

Λ	Ω	δ_2/δ	S	H	λ	Ω_2	D_1
16.00	-4.41750	0.10244	0.28737	2.40670	0.16790	-0.45253	0.18569
14.00	-3.19130	0.10039	0.29769	2.38050	0.14109	-0.32037	0.18816
12.00	-1.91800	0.09776	0.31113	2.34782	0.11469	-0.18751	0.19147
10.00	-0.60000	0.09464	0.32899	2.30569	0.08957	-0.05679	0.19621
8.00	0.91609	0.08930	0.35318	2.25507	0.06379	0.08180	0.20249
6.00	2.64000	0.08208	0.38607	2.19006	0.04042	0.21669	0.21191

Table 3-6b Distribution in integral parameters for plane stagnation flow.

Λ	BP	FF1	a_4	α_4	RESR
16.00	1.10437	0.70131	-2.02059	11.19175	0.06087
14.00	0.85290	0.79250	-1.50739	10.15668	0.05766
12.00	0.55368	0.91873	-1.03744	9.09978	0.05246
10.00	0.18953	1.09804	-0.64104	8.01230	0.04425
8.00	-0.32423	1.39983	-0.28240	6.93151	0.03533
6.00	-1.07771	1.92011	-0.04055	5.83754	0.02344

factor $(f_x/2)\sqrt{Re_x}$ is plotted against BP in Fig. 3-10 and the solution is compared with similarity solution. The accuracy obtained is within 1%.

The calculations for velocity distribution are shown in Fig. 3-12.

3.4.2.1.3 Axisymmetric Stagnation Flow

For axisymmetric stagnation flow

$$(2\lambda - F_2) = 0 \tag{3.21a}$$

and

$$\frac{f_x}{2}\sqrt{Re_x} = \sqrt{2/F_2}\ S \tag{3.21b}$$

Calculations for the integral parameters for transpired axisymmetric flow are given in Table. 3-7. The friction factor $(f_x/2)\sqrt{Re_x}$ is plotted against BP in Fig. 3-10 and the solution is compared with similarity solution. The accuracy obtained is within 1%.

The calculations for velocity distribution are shown in Fig. 3-13.

3.4.2.2 Nonsimilar Flows

To test the composite method developed for nonsimilar transpired flow, consideration is given to asymptotic suction flow.

3.4.2.2.1 Asymptotic Suction Flow with Uniform Free Stream Velocity

Asymptotic suction boundary layer flows are associated with uniform negative values of the dimensionless transpiration rate v_0/U_∞.

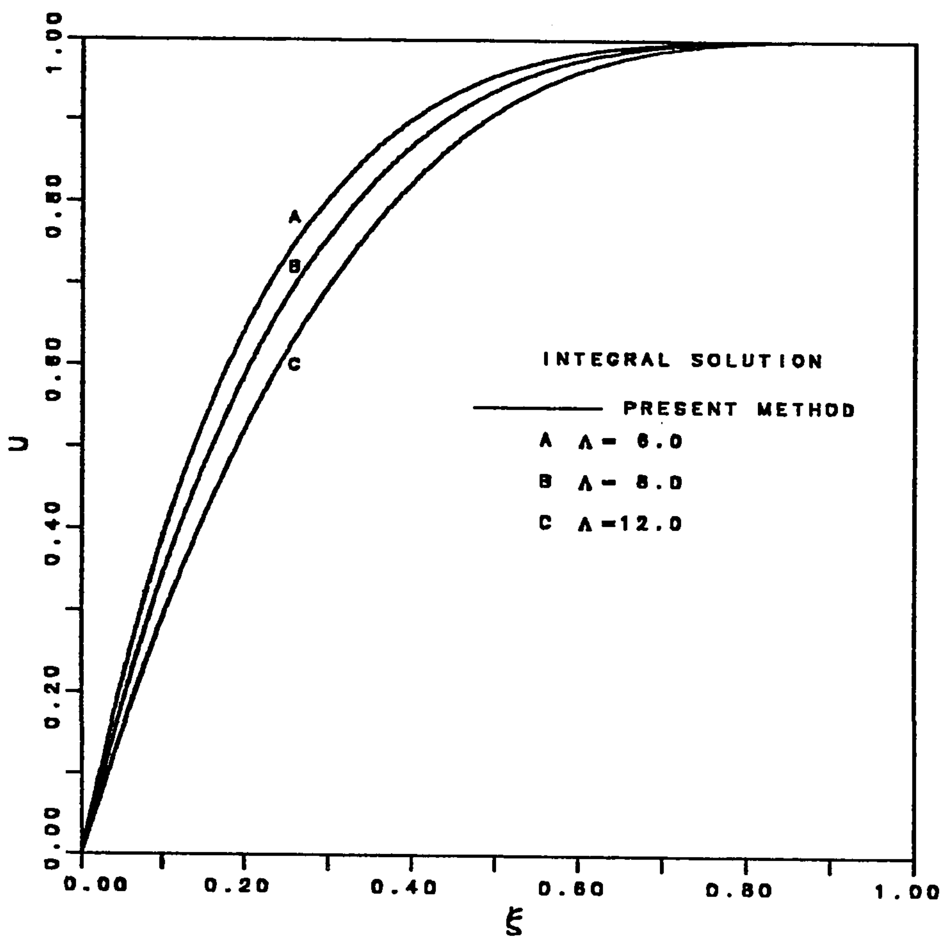

Fig. 3-12 Approximations for velocity distribution for plane stagnation flow.

Table 3-7a Distribution in integral parameters for axisymmetric stagnation flow.

Λ	Ω	δ_2/δ	S	H	λ	Ω_2	F_2
12.00	-4.46700	0.10508	0.25684	2.48074	0.13251	-0.46941	0.26501
10.00	-2.92650	0.10261	0.27239	2.43911	0.10529	-0.30029	0.21058
8.00	-1.30500	0.09916	0.29415	2.38463	0.07866	-0.12940	0.15734
7.00	-0.51000	0.09791	0.30881	2.34649	0.06710	-0.04993	0.13418
5.00	1.44300	0.09051	0.34643	2.26851	0.04096	0.13061	0.08191
4.00	2.53530	0.08587	0.37164	2.21847	0.02950	0.21772	0.05899
2.00	5.49600	0.07039	0.43740	2.10094	0.00991	0.38686	0.01982

Table 3-7b Distribution in integral parameters for axisymmetric stagnation flow.

Λ	BP	D_1	FF1	a_4	α_4	RESR
12.00	1.28953	0.17969	0.70558	-3.57320	6.28925	-0.02688
10.00	0.92543	0.18297	0.83944	-2.54023	5.98660	-0.00618
8.00	0.46136	0.18782	1.04875	-1.62296	5.59899	0.00873
7.00	0.19277	0.19183	1.19277	-1.26575	5.31868	0.01450
5.00	-0.64542	0.20074	1.71184	-0.56249	4.82453	0.01665
4.00	-1.26766	0.20760	2.16392	-0.31830	4.51393	0.01505
2.00	-3.88633	0.22756	4.39411	-0.02483	3.83530	0.00690

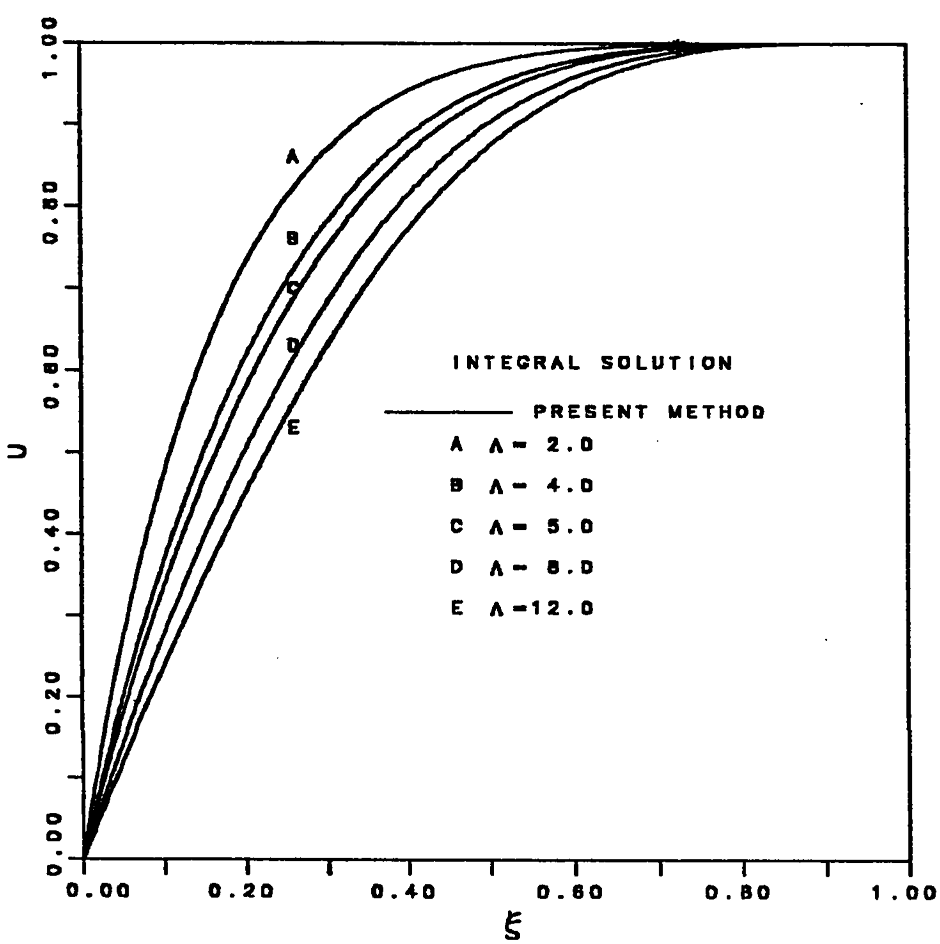

Fig. 3-13 Approximations for velocity distribution for axisymmetric stagnation flow.

As x increases the friction factor asymptotically approaches the dimensionless transpiration rate,

$$\frac{f_x}{2} = -\frac{v_0}{U_\infty} \tag{3.22}$$

Calculations for friction factors obtained by the use of composite one and two-parameter integral method developed is plotted against BP in Fig. 3-14. Integral results obtained with one-parameter method by using N = 2 and 3 are also shown. The accuracy obtained is within 1%.

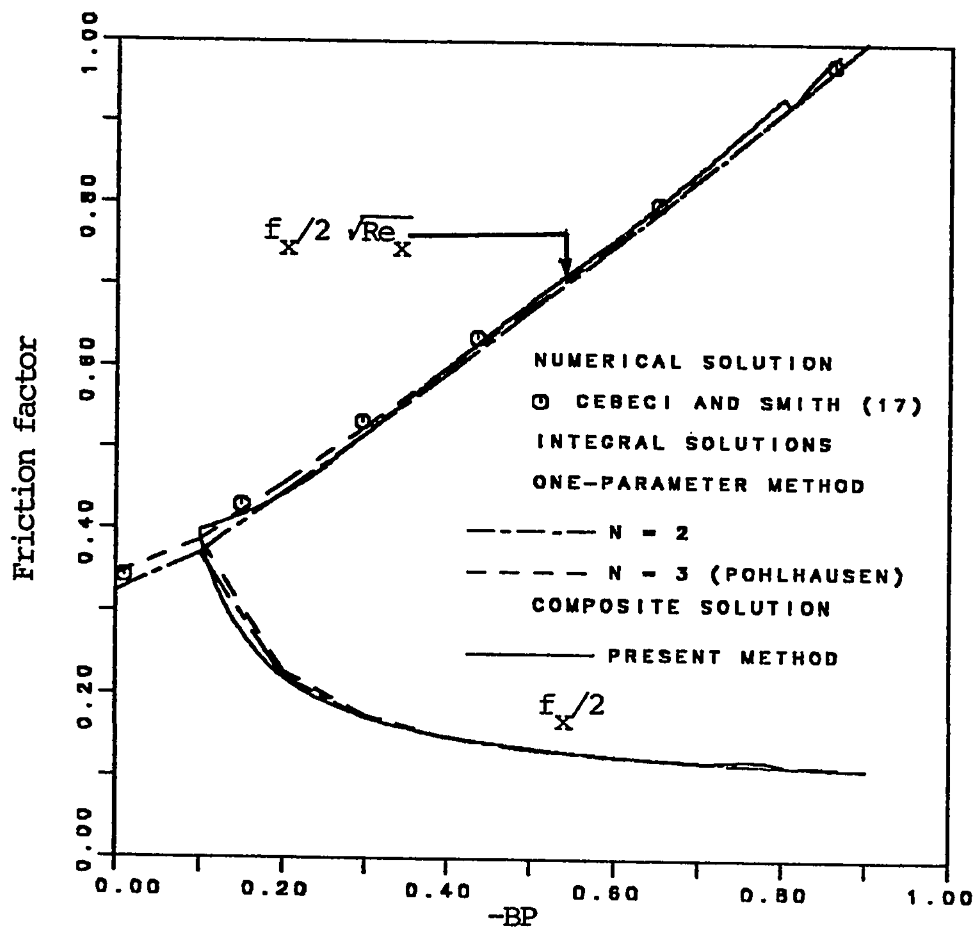

Fig. 3-14 Calculations for friction factor for nonsimilar boundary layer flow with uniform suction and uniform free stream velocity.

4. CONCLUSIONS AND RECOMMENDATIONS

Integral methods of the first and second kind are available in the literature for analyzing boundary layer flow. Whereas integral methods of the first kind feature the use of supplementary boundary layer approximations for velocity or stress in terms of the dimensionless distance from the wall, the methods of the second kind are characterized by approximations for viscous stress in terms of velocity.

Although integral methods of the first kind bear good recognition, these methods have undergone little development over the past twenty years. One and two-parameter integral methods of the first kind have been constantly used in the analysis of nontranspired laminar boundary layer flow with moderate favorable pressure gradients to separation. Integral methods of this kind have also been effectively applied to heat transfer and natural convection flow, and provide a fundamental basis for generalization to transpired and turbulent flow. The adaptation of this method to laminar transpired flow has only recently been accomplished [3]. This approach is generally very practical and relatively simple.

On the other hand, integral methods of the second kind have undergone rather intensive development in recent years. Accurate multiple parameter integral methods of the second kind have been developed for laminar and turbulent forced convection flow with and without transpiration. However, these methods generally require the use of numerous parameters and have been developed in the context of rather intimidating forms of higher order integral equations. Furthermore, because of the nature of the approximation used for stress, methods of the second kind are not applicable to natural convection flow.

The objective of this thesis is to develop a practical and reliable integral method for transpired laminar boundary layer flow. Because of its relative simplicity and practicability, the method of the first kind has been selected.

In the integral method of the first kind, the viscous stress τ_{xy} has been approximated by Eq.(2.28a),

$$\frac{\tau_{xy}}{\tau_s} = \sum_{n=0}^{N} a_n \xi^n$$

for nontranspired flow, and by Eq.(2.28b),

$$\frac{\tau_{xy}}{\tau_s} = \sum_{n=0}^{N} a_n \xi^n + B_m U$$

for transpired flow. Using these approximations for viscous stress, relations have been developed for velocity distribution of the form of Eq.(2.24),

$$U = \sum_{n=1}^{N+1} C_n \xi^n$$

for nontranspired flow, and Eq.(2.35),

$$U = \sum_{n=0}^{N} C_n \xi^n - C_0 e^{-\Omega\xi}$$

for transpired flow [3]. These relations have been used to develop one and two-parameter integral methods for laminar boundary layer flow. One-parameter methods provide an accuracy of 3%, except in the vicinity of separation where the error can trigger to 10 to 15% and are very efficient. A complementary 4th order two-parameter method of this kind ascertains an accuracy of within about 1% for adverse pressure gradient up to separation, but breaks down for mild favorable pressure gradients and suction.

In order to develop a reliable integral method of the first kind which is applicable to a wide range of conditions, various methods of characterizing flows with strong favorable pressure gradient and suction were considered. A 4th order one-parameter integral method was found to be the most practical approach from the standpoint of range, reliability and simplicity. Therefore, a composite one and two-parameter 4th order integral method of the first kind has been developed in this thesis for laminar transpired and nontranspired boundary layer flow. The method operates as a one-parameter method for strong favorable pressure gradient and strong suction, and as a two-parameter method for mild to adverse pressure gradient and blowing.

Using the one and two-parameter 4th order velocity distribution given by Eqs.(2.24) and (2.35), relations have been developed for δ_1/δ, δ_2/δ, δ_3/δ, D_1 and related integral parameters, with the parameters S, λ, H, Ω_2 and F_2 being calculated by nested do loops. Using these relations, the integral momentum equation, Eq.(2.8), and integral mechanical energy equation, Eq.(2.13), have been solved to obtain solutions for similar and nonsimilar boundary layer flows. In the one-parameter method δ is the only unknown parameter, whereas in the two-parameter method δ and a_4 constitute the two unknown parameters. The second unknown parameter a_4 is evaluated by iterating on α_4 in the context of solving the integral mechanical energy equation. The Newton-Raphson method has been used in the iterative calculations.

The resulting composite integral method has been tested for a wide range of flow conditions. The method has been applied to both transpired and nontranspired, similar and nonsimilar flows. For similar nontranspired flow, separation is indicated at $\beta = -0.20083$ ($\Lambda = -5.23098$), with the method being applicable for values of β as large as 10 ($\Lambda \le 20.0$). For similar transpired flow, the method operates in the range $-0.63889 \le \Omega_2 \le 0.49999$ ($-5.889 \le \Omega \le 20.0$), with separation occuring at $\Omega_2 = -0.63889$ ($\Omega = -5.889$). The results obtained for friction factor and velocity distributions are compared with exact similarity solution. The method generally gives an accuracy of within 1%. Better results have been observed for both plane and axisymmetric stagnation flows.

The method has also been tested for nonsimilar boundary layer flow. Because of the presence of nonlinear term in the integral mechanical energy equation, Eq (2.57), a simple numerical finite difference method has been used to obtain integral solution for nonsimilar boundary layer flow. Linear retarded and accelerated flow, plane flow over a circular cylinder, axisymmetric flow over a sphere and asymptotic suction flow with uniform free stream velocity cases have been considered to demonstrate the usefulness of the integral method developed for transpired and nontranspired nonsimilar boundary layer flows. The results were compared with the numerical calculations and an accuracy of within 1 to 2% was obtained for these flows.

4.1 INTEGRAL METHODS FOR TURBULENT BOUNDARY LAYER FLOW

One of the primary motivations in undertaking this work was to establish a framework for the development of a practical integral method for turbulent boundary layer flow.

Turbulent flows are characterized by random fluctuating flow. The integral equations and integral relations for turbulent boundary layer flow are of the same form as those for laminar boundary layer flow, except that the transport characteristics u and v are replaced by the mean characteristic $\bar{u}$ and $\bar{v}$ and viscous stress τ_{xy} is replaced by the total stress $\bar{\tau}$. The total stress $\bar{\tau}$ can be written as

$$\bar{\tau} = \bar{\tau}_{xy} + \bar{\tau}_t \tag{4.1}$$

$$= \bar{\tau}_{xy} - \rho\overline{u'v'} \tag{4.2}$$

where $\bar{\tau}_{xy}$ is the mean viscous stress and $\bar{\tau}_t$ $(= -\rho\overline{u'v'})$ is the Reynolds stress. The Reynolds stress $\bar{\tau}_t$ is generally expressed in terms of the mean velocity $\bar{u}$ by

$$\bar{\tau}_t = \mu_t \frac{\partial \bar{u}}{\partial y} \tag{4.3}$$

where μ_t is the turbulent viscosity, or

$$\bar{\tau}_t = \rho \iota^2 \left(\frac{\partial \bar{u}}{\partial y}\right)^2 \tag{4.4}$$

where ι is the mixing length. It also follows that the total stress can be represented by

$$\begin{aligned} \bar{\tau} &= \frac{\partial \bar{u}}{\partial y}(\mu + \mu_t) \\ &= \mu \frac{\partial \bar{u}}{\partial y} + \rho \iota^2 \left(\frac{\partial \bar{u}}{\partial y}\right)^2 \end{aligned} \tag{4.5}$$

The mixing length within the important inner region can be approximated by the relation given by Van Driest [22]

$$\iota^+ = \kappa y^+ D \tag{4.6}$$

where $y^+ = yU^*/\nu$, $\iota^+ = \iota U^*/\nu$, $U^* = \sqrt{\bar{\tau}_0/\rho}$, the damping factor D is given by

$$D = 1 - \exp\left(-\frac{y^+}{A^+}\right) \tag{4.7}$$

and A^+ is the damping parameter; $\kappa = 0.41$ as recommended by Von Karman [23]. The damping parameter A^+ can be calculated by using the following empirical relation developed by Kays [24] for near equilibrium flow with moderate pressure gradient and rate of transpiration:

$$A^+ = \frac{25}{a[v_0^+ + b(\frac{P^+}{1 + cv_0^+})] + 1} \tag{4.8}$$

where

$b = 4.25, \quad c = 10 \quad$ for $P^+ < 0$

$b = 2.90, \quad c = 0 \quad$ for $P^+ > 0$

$a = 7.10, \quad$ for $v_0^+ \geq 0$

$a = 9.00, \quad$ for $v_0^+ < 0$

As y^+ increases the damping factor approaches unity such that Eq.(4.6) reduces to

$$\iota^+ = \kappa y^+ \tag{4.9}$$

which is applicable in the intermediate region. The mixing length for the outer region is approximated by

$$\iota^+ = \alpha_0 \delta^+ \tag{4.10}$$

where $\delta^+ = \delta U^*/\nu$. Kays and Anderson [25] suggested the following relation for α_0

$$\alpha_0 = 0.0779(6000/Re_{\delta_2})^{1/8}(1 - 67.5\ F) \quad \text{for } Re_{\delta_2} \leq 6000 \tag{4.11a}$$

$$\alpha_0 = 0.0779(1 - 67.5\ F) \quad \text{for } Re_{\delta_2} > 6000 \tag{4.11b}$$

where F ($\equiv v_0/U_\infty$) is known as the blowing fraction.

The mean velocity $\bar{u}$ is expressed in terms of $\bar{\tau}$, mixing length ι and wall variables u^+ and y^+ by the relation

$$\frac{\bar{\tau}}{\bar{\tau}_0} = \frac{\partial u^+}{\partial y^+} + \iota^{+2}\left(\frac{\partial u^+}{\partial y^+}\right)^2 \tag{4.12}$$

This equation is solved to obtain u^+ as

$$u^+ = 2\int_0^{y^+} \frac{\bar{\tau}/\bar{\tau}_0 \, dy^+}{1 + \sqrt{1 + 4\iota^{+2}\, \bar{\tau}/\bar{\tau}_0}} \tag{4.13}$$

Focusing attention on the integral formulation, the expressions for δ_1, δ_2, δ_3 and δ_4 in terms of U ($\equiv \bar{u}/U_\infty$) are identical for laminar and turbulent flow. However, the Reynolds stress occurs in the dissipation term which appear in the higher order integral equations; for example,

$$\mathcal{D} = \int_0^\infty \bar{\tau}\frac{\partial \bar{u}}{\partial y} dy \tag{4.14}$$

In the following section integral methods of first and second kind for turbulent boundary layer flow will be discussed.

4.1.1 Integral Method of the First Kind

The integral method of the first kind which is featured in this study is being adapted to turbulent boundary layer flow at KFUPM in a parallel study [26]. Supplementary boundary layer approximations for the total stress $\bar{\tau}$ have been assumed to be of the same form as laminar boundary layer flow. Hence $\bar{\tau}$ can be approximated by

$$\frac{\bar{\tau}}{\tau_0} = \sum_{n-0}^{N} a_n \xi^n + B_m \frac{\bar{u}}{U_\infty} \tag{4.15}$$

where

$\xi = y/\delta$, $B_m = \rho\bar{v}_0 U_\infty/\bar{\tau}_0 = v_0^+ U_\infty^+$, $v_0^+ = \bar{v}_0/u^*$, $U_\infty^+ = U_\infty/u^* = \sqrt{2/f_x}$.

Using the Couette law,

$$\frac{\bar{\tau}}{\bar{\tau}_0} = 1 + \beta_\delta\xi + B_m\bar{u}/U_\infty = 1 + P^+y^+ + v_0^+u^+ \qquad (4.16)$$

near the wall and the constraints

$$\frac{\partial\bar{u}}{\partial y} = 0 \quad \text{and} \quad \bar{\tau} = 0 \qquad (4.17a)$$

$$\frac{\partial\bar{u}}{\partial y} = 0 \quad \text{and} \quad \bar{u} = U_\infty \quad \text{at} \quad y = \delta \qquad (4.17b)$$

and setting N = 3, Eq.(4.15) has been solved to obtain a one-parameter 3rd order approximation of the form

$$\frac{\bar{\tau}}{\bar{\tau}_0} = 1 + \beta_\delta\xi + B_m\frac{\bar{u}}{U_\infty} - (3 + 2\beta_\delta + 3B_m)\xi^2$$

$$+ (2 + \beta_\delta + 2B_m)\xi^3 \qquad (4.18)$$

or, in terms of wall variables,

$$\frac{\bar{\tau}}{\bar{\tau}_0} = 1 + P^+y^+ + v_0^+u^+ - (3 + 2P^+\delta^+ + 3v_0^+U_\infty^+)\xi^2$$

$$+ (2 + P^+\delta^+ + 2v_0^+U_\infty^+)\xi^3 \qquad (4.19)$$

where $\beta_\delta = \delta/\bar{\tau}_0(d\bar{P}/dx) = \delta^+P^+$, $y^+ = yu^*/\nu$, $P^+ = \nu/(u^{*3}\rho)d\bar{P}/dx$

This expression has been used to develop a one-parameter integral method for turbulent boundary layer flow. With $\bar{\tau}$ given by Eq.(4.19) and ι given by Eqs.(4.6), (4.9) and (4.10), Eq.(4.13) can be numerically integrated to compute the distribution in u^+ across the boundary layer. Consistent results have been obtained for mild to moderate pressure gradients with transpiration. The method is simple and very practical.

4.1.2 Integral Methods of the Second Kind

Integral method of the second kind have been developed for turbulent boundary layer flow by Murphy and Rose [27], Abott and Deiwert [28], Yeung and Yang [29] and Fletcher and Fleet [30]. These approaches are characterized by the use of approximations for viscous stress of the general form of Eq.(1.3),

$$\theta = \frac{\rho U_\infty^2}{\tau_{xy}} = \frac{1}{1-U} \sum_{j=1}^{N-1} A_j W_j(U) \tag{4.20}$$

where A_j represents N unspecified parameters and W_j is a weighting function. The integral momentum equation and N-1 higher order integral equations are used to evaluate the N parameters A_j. Using Eq.(4.20), relations are readily obtained for yU_∞/ν, $\delta_1 U_\infty/\nu$, $\delta_2 U_\infty/\nu$ and other integral thicknesses which are of the same form for laminar and turbulent flow. The distinction between laminar and turbulent flow is accounted for in these approaches by the dissipation terms which appear in the higher order integral equations. For example, the dissipation term $\mathcal{D}$ which shows up in the integral mechanical energy equation is expressed in terms of $\bar{\theta}$ by writing

$$\frac{\mathcal{D}_1}{\rho U_\infty^3} = \int_0^1 (1 + \frac{\mu_t}{\mu})\frac{1}{\theta}\, dU \tag{4.21a}$$

$$= \int_0^1 [\frac{1}{\theta} + (\frac{U_\infty \iota}{\nu})^2 \frac{1}{\theta^2}] dU \tag{4.21b}$$

With the mixing length ι specified by Eqs.(4.6), (4.9) and (4.10), $\mathcal{D}_1/\rho U_\infty^3$ and higher order dissipation terms can be calculated by numerical integration.

To start the integral solution, the parameters A_j must be specified at the first station. This is done by using experimental data for the velocity distribution or inputs for f_x and the integral thicknesses.

Simple two and three- parameter methods of the second kind are generally incapable of characterizing the velocity distribution for turbulent boundary layer flow. Therefore the order and complexity of these approaches for analyzing turbulent boundary layer flow have increased reasonably over the past several years. To achieve computationally efficient method which is capable of operating with a sufficient number of parameters, Yeung and Yang [29] proposed the use of an orthonormal approximation for θ of the form of Eq.(4.20) with

$$W_i = \sum_{k-1}^{i} C_{ik}\,(1-U)^k \tag{4.22}$$

and

$$\int_0^1 W_k W_j \frac{U}{1-U} dU = 1 \quad j = k$$

$$= 0 \quad j \neq k \tag{4.23}$$

Using this distribution, the number of parameters appearing in each integral equation reduce from N to two, such that the resulting system of equations is more efficiently solved. Yeung and Yang developed three, four and five-parameter calculations for adverse, favorable and zero pressure gradient flows. Although the method provides reasonable accuracy for flow with moderate favorable pressure gradient, it has proven to be unreliable for adverse pressure gradient flow.

The most recent work along this line has been reported by Fletcher and Fleet [30]. Using a somewhat different form of the integral equations together with multiple parameters, Fletcher and Fleet have achieved good results for a range of turbulent transpired boundary layer flows. However, the method is quite involved and is not suitable for natural convection flows.

From the above discussion, it can be concluded that a simple, lower order integral method of the second kind for turbulent boundary layer flow is not possible. In order to have an integral method of the second kind for turbulent boundary layer flow, higher order multiple parameter approximations are required, which make the method extremely involved.

Because of their relative simplicity and practicability, it is strongly recommended that significant attention be directed to the integral methods of the first kind in future development work for turbulent boundary layer flow.

NOMENCLATURE

A	damping parameter
A^+	$= AU^*/\nu$, see Eq.(4.8)
A_0, A_1, A_2	coefficients used in Eq.(2.67)
a	constant in Eq.(4.8)
a_0, a_1, a_2, a_3, a_4	coefficients used in Eq.(2.29)
B_m	$= \rho v_0 U_\infty/\tau_0$, blowing parameter
BP	$= v_0/U_\infty \sqrt{Re_x}$
b	constant in Eq.(4.8)
c	constant in Eq.(4.8)
C	constant in Eq.(2.20)
C_0, C_1, C_2, C_3, C_4	coefficients used in Eq.(2.35)
D	damping factor
$\mathcal{D}$	dissipation integral
D_1	$= \mathcal{D}_1 \delta_2/(\mu U_\infty^2)$
D_2	$= \mathcal{D}_2 \delta_2/(\mu U_\infty^2)$
F	$= v_0/U_\infty$, blowing fraction
F_2	$= 1/r_0^2 \; U_\infty/\nu \; d/dx \; (r_0\delta_2)^2$
F_3	$= 1/r_0^2 \; U_\infty/\nu \; d/dx \; (r_0\delta_3)^2$
f_x	$= \tau_0/(\rho U_\infty^2/2)$, Fanning friction factor
FF1	$= (f_x/2)\sqrt{Re_x}$, friction factor
G_2	$= U_\infty/\nu \; d\delta/dx/L$

H	$= \delta_1/\delta_2$, shape factor
H_{31}	$= \delta_3/\delta_1$
H_{32}	$= \delta_3/\delta_2$
K	$= (\nu/U_\infty^2)\, dP/dx$, acceleration parameter
L	characteristic length
ι	mixing length
ι^+	$= \iota U^*/\nu$
Mo_δ	$= \delta\tau_0/(\mu U_\infty)$
M_0, M_1, M_2, M_3	see Eq. (2.36)
m	$= (x/U_\infty)\, dU_\infty/dx$, pressure gradient parameter
P	pressure
P^+	$= (\nu/\rho U^{*3})\, d\overline{P}/dx$
R	residue
Re_x	$= U_\infty x/\nu$, Reynolds number based on x
r_0	radius of curvature
S	$= \tau_0\delta_2/(\mu U_\infty)$
U	$= u/U_\infty$, dimensionless velocity
u_0	orbitrary constant intial velocity
U_∞	free stream velocity
U_∞^+	$= U_\infty/U^*$
u	instantaneous velocity in x-direction
u_1	reference velocity

$\overline{u}$	mean velocity in x-direction
u'	velocity fluctuations in x-direction
u^*	$= \sqrt{\overline{\tau}_0/\rho}$, friction velocity
v	instantaneous velocity in y-direction
$\overline{v}$	mean velocity in y-direction
v'	velocity fluctuations in y-direction
v_0	transpiration velocity
$\overline{v}_0$	mean transpiration velocity
v_0^+	$= \overline{v}_0/U^*$
W_j	weighting function
x, y	co-ordinates paralell and normal to the wall
x_1	reference axial location
Δx	reference axial distance
y^+	$= yU^*/\nu$

Greek Symbols

α_0	mixing length parameter for outer region
α_4	iterative parameter
β	Falkner-Skan acceleration parameter
β_4	constant
β_δ	$= (\delta/\tau_0)\, dP/dx$
γ_4	constant

δ	boundary layer thickness
δ^+	$= \delta U^*/\nu$, see Eq.(4.10)
δ_1	displacement thickness
δ_2	momentum thickness
δ_3	kinetic energy thickness
δ_4	dissipation thickness
η	$= y\sqrt{U_\infty/\nu x}$, normal transformed co-ordinate
Λ	$= (\delta^2/\nu)\, dU_\infty/dx$
λ	$= (\delta_2^2/\nu)\, dU_\infty/dx$
μ	dynamic viscosity
μ_t	turbulent viscosity
ν	$= \mu/\rho$, kinematic viscosity
ρ	density
ξ	$= y/\delta$
τ_0	wall shear stress
$\bar{\tau}_0$	mean wall shear stress
τ_{xx}	viscous normal stress
τ_{xy}	viscous shear stress
$\bar{\tau}_{xy}$	mean viscous shear stress
$\bar{\tau}$	mean total shear stress
$\bar{\tau}_t$	Reynolds Stress

Ω	$= -v_0\delta/\nu$
Ω_2	$= -v_0\delta_2/\nu$
θ	$= \rho U_\infty^2/\tau_{xy}$
θ_0	$= \rho U_\infty^2/\tau_0$
κ	constant (= 0.41)

<u>Subscripts</u>

i	constant property condition at $x = x_i$
i+1	constant property condition at $x = x_{i+1}$

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